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Non-minimal order low-frequency $\mathscr{H}_\infty$ filtering for uncertain discrete-time systems

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Abstract: This paper proposes a sufficient condition for the discrete-time robust $\mathscr{H}_\infty$ filtering design problem with low-frequency specifications using an extension of the generalized Kalman-Yakubovich-Popov lemma. The matrices of the system are supposed to be uncertain, time-invariant and to belong to a polytopic domain. The proposed approach takes advantage of a non-minimal filter structure, that is, a filter with order greater than the order of the system being filtered, to provide improved $\mathscr{H}_\infty$ bounds for low-frequency specifications. The condition can be solved by means of linear matrix inequality relaxations with slack variables and Lyapunov matrices which are considered as homogeneous polynomials of arbitrary degree. Numerical examples illustrate the improvements on the $\mathscr{H}_\infty$ bounds provided by the non-minimal filter structure in combination with the more accurate polynomial approximations (higher degrees) for the optimization variables.

Keywords: Robust linear matrix inequalities, Filtering, Generalized Kalman-Yakubovich-Popov Lemma, Low-frequency Specifications, Linear time-invariant uncertain systems.

1. INTRODUCTION

The significance and utility of the Kalman-Yakubovich-Popov (KYP) lemma in control theory and related areas are well recognized. The KYP lemma encompasses other well known results as special cases, such as positive-realness and characterization of boundedness of rational transfer functions. The former result is essential for linear estimation, assuring the existence of a stabilizing positive-definite solution for a Riccati equation. This stabilizing solution is then used to obtain the spectral factorization of the transfer matrix (Kailath et al., 2000). The latter result is the so-called Bounded Real Lemma, which is widely used in the context of $\mathscr{H}_\infty$ control. An algebraic proof for the KYP lemma can be found in Rantzer (1996).

The KYP was extended to deal with finite-frequency specifications by Iwasaki and Hara (2005), in a result known as the generalized KYP lemma (gKYP). This extension relates frequency-domain inequalities to semidefinite constraints, being useful to ameliorate the $\mathscr{H}_\infty$ performance of control systems that operate in specific frequency ranges. See Graham et al. (2009) for another equivalent extension of the gKYP.

It is worth to point out some important and recent results regarding the development of the gKYP theory. First, the paper of Pipeleers and Vandenbergue (2011) has furthered into the theory by proving realness of some matrices in the formulation. Second, in opposition to the original article of Iwasaki and Hara (2005) that used S-procedure, the paper of You and Doyle (2013) has shown the importance of Lagrange duality to enhance the understanding of the gKYP.

Although the gKYP lemma is necessary and sufficient for analysis purposes, devising exact conditions for the design of filters and controllers is, to the best of the authors' knowledge, an open problem in the control theory. For continuous-time systems and middle-frequency specifications, a necessary and sufficient condition for control and estimation problems with complex realization was proposed in ao et al. (2016).

Following similar steps, this paper addresses the problem of robust low-frequency $\mathscr{H}_\infty$ filter design for uncertain discrete-time systems using the gKYP formulation. Sufficient parameter-dependent linear matrix inequality (LMI) conditions are given for the existence of asymptotically stable filters that fulfill a low-frequency specification. The proposed approach takes advantage of a non-minimal order structure for the filter, that is, the order of the designed filter is greater than the order of the plant being filtered, which may provide less conservative bounds when dealing with uncertain linear time-invariant (LTI) systems. The non-minimal structure follows the lines of the works of Lee and Joo (2014); Frezzatto et al. (2015, 2016, 2017) that use this filter structure in lessening the conservatism of design conditions as the filter order enlarges. As shown in Frezzatto et al. (2016), the performance of such filters in terms of the $\mathscr{H}_\infty$ norm criterion cannot deteriorate with the growth of the filter order. Numerical results are presented to show the effectiveness of the proposed approach.

The rest of the paper is organized as follows. Section 2 states some preliminary results. A detailed description of the prob-

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lem is provided in Section 3, and the main contributions are presented in Section 4. Section 5 provides some numerical examples illustrating the effectiveness of the proposed approach. Finally, Section 6 concludes the paper.

The notation used throughout this paper is standard. The symbol $(^T)$ indicates the transpose of matrices or vectors. The operator $\mathrm{He}(A) = A + A^T$ is used to shorten formulas. The set of symmetric matrices of dimension n is denoted by $\mathbf{S}_n$. For symmetric matrices, $A \succ 0$ ($A \prec 0$) means that A is positive (negative) definite. A symmetric term in a block matrix is denoted by $\star$. The identity matrix is denoted by I and the zero matrix by 0. The set of natural numbers is denoted by $\mathbb{N}$.

2. PRELIMINARIES

The well-known generalized KYP lemma (Iwasaki and Hara, 2005) is essential for the development presented in this section, especially the version for discrete-time low-frequency range. This result is reproduced in the next lemma.

Lemma 1. Let matrices $A \in \mathbb{R}^{n\times n}$ with no eigenvalues on the unit circle, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times r}$, and a scalar v_ℓ be given. The following statements are equivalent:

i) $\|H(\zeta)\|_\infty < \gamma$, $\forall \zeta = e^{j\omega}$ and $|\omega| < v_\ell$, where
$$H(\zeta) = \mathsf{C}(\zeta I - \mathsf{A})^{-1}\mathsf{B} + \mathsf{D}.$$

ii) There exist matrices $P, Q \in \mathbf{S}_n$, $Q \succ 0$, such that
$$\begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -P & Q \\ Q & P - 2\cos(v_\ell)Q \end{bmatrix} \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \mathsf{C} & \mathsf{D} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \mathsf{C} & \mathsf{D} \\ 0 & I \end{bmatrix} \prec 0. \quad (1)$$

Proof. The proof is presented in Iwasaki and Hara (2005).

Although condition (1) is necessary and sufficient for analysis purposes, the products involving the matrices of the system (particularly, A and B) and variable P do not allow an immediate extension to cope with synthesis of controllers and filters. Hence, to surmount this inconvenience, a necessary and sufficient condition with three extra multipliers is proposed in the next lemma.

Lemma 2. Let matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times r}$, and a scalar v_ℓ be given. Then, the following condition is equivalent to (1):

i) There exist matrices $P, Q \in \mathbf{S}_n$, $Q \succ 0$, and matrices $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{p\times n}$ such that
$$\begin{bmatrix} -\mathsf{P} - \mathrm{He}(F) & \mathscr{P}_{12} & F\mathsf{B} - H^T & 0 \\ \star & \mathscr{P}_{22} & G\mathsf{B} + \mathsf{A}^T H^T & \mathsf{C}^T \\ \star & \star & -\gamma^2 I + \mathrm{He}(H\mathsf{B}) & \mathsf{D}^T \\ \star & \star & \star & -I \end{bmatrix} \prec 0. \quad (2)$$
where
$$\mathscr{P}_{12} = \mathsf{Q} + F\mathsf{A} - G^T, \quad \mathscr{P}_{22} = P - 2\cos(v_\ell)\mathsf{Q} + \mathrm{He}(G\mathsf{A}).$$

Proof. First, by applying a Schur complement with respect to the $(4,4)$ block, inequality (2) can be rewritten as
$$\mathscr{Q} + \begin{bmatrix} F \\ G \\ H \end{bmatrix} \begin{bmatrix} -I & \mathsf{A} & \mathsf{B} \end{bmatrix} + \begin{bmatrix} -I \\ \mathsf{A}^T \\ \mathsf{B}^T \end{bmatrix} \begin{bmatrix} F^T & G^T & H^T \end{bmatrix} \prec 0, \quad (3)$$
with
$$\mathscr{Q} = \begin{bmatrix} -P & Q & 0 \\ \star & P - 2\cos(v_\ell)Q + \mathsf{C}^T\mathsf{C} & \mathsf{C}^T\mathsf{D} \\ \star & \star & \mathsf{D}^T\mathsf{D} - \gamma^2 I \end{bmatrix}.$$
Then, using Finsler's Lemma (de Oliveira and Skelton, 2001), inequality (3) holds if, and only if,
$$\begin{bmatrix} \mathsf{A}^T & I & 0 \\ \mathsf{B}^T & 0 & I \end{bmatrix} \mathscr{Q} \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \\ 0 & I \end{bmatrix} \prec 0,$$
which is, indeed, inequality (1).

3. PROBLEM DEFINITION

Consider an asymptotically stable uncertain discrete LTI system given as
$$\begin{aligned} x(k+1) &= A(\xi)x(k) + B_w(\xi)w(k) \\ y(k) &= C_y(\xi)x(k) + D_{yw}(\xi)w(k) \\ z(k) &= C_z(\xi)x(k) + D_{zw}(\xi)w(k) \end{aligned} \quad (4)$$
where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^r$ is the noise input, $z \in \mathbb{R}^p$ is the output to be estimated, and $y \in \mathbb{R}^q$ represents the measured output. The matrices of the system
$$U(\xi) = \begin{bmatrix} A(\xi) & B_w(\xi) \\ C_y(\xi) & D_{yw}(\xi) \\ C_z(\xi) & D_{zw}(\xi) \end{bmatrix},$$
belong to the polytopic set given by
$$\mathscr{X} := \left\{ U(\xi) \, : \, U(\xi) = \sum_{i=1}^{N} \xi_i U_i, \; \xi \in \Xi_N \right\},$$
where
$$\Xi := \left\{ \xi \in \mathbb{R}^N \, : \, \sum_{i=1}^{N} \xi_i = 1, \; \xi_i \geq 0 \right\},$$
is the unit simplex of dimension N. The goal is to design an asymptotically stable robust filter with a non-minimal realization whose state-space representation is
$$\begin{aligned} x_f(k+1) &= A_f x(k) + B_f y(k) \\ z_f(k) &= C_f x(k) + D_f y(k) \end{aligned} \quad (5)$$
where $x_f \in \mathbb{R}^{n_f}$, $n_f = n + m$, $m \in \mathbb{N}$, that minimizes a bound to the $\mathscr{H}_\infty$ norm in low-frequency range of the transfer matrix associated with the dynamic of the error, $e(k) = z(k) - z_f(k)$, that is given by
$$H(\zeta, \xi) = \mathsf{C}(\xi)(\zeta I - \mathsf{A}(\xi))^{-1}\mathsf{B}(\xi) + \mathsf{D}(\xi), \quad \forall \xi \in \Xi \quad (6)$$
with
$$\left[\begin{array}{c} \mathsf{A}(\xi) \\ \hline \mathsf{C}(\xi) \end{array}\right] = \left[\begin{array}{cc} A(\xi) & 0 \\ B_f C_y(\xi) & A_f \\ \hline C_z(\xi) - D_f C_y(\xi) & -C_f \end{array}\right],$$
$$\left[\begin{array}{c} \mathsf{B}(\xi) \\ \hline \mathsf{D}(\xi) \end{array}\right] = \left[\begin{array}{c} B_w(\xi) \\ B_f D_{yw}(\xi) \\ \hline D_{zw}(\xi) - D_f D_{yw}(\xi) \end{array}\right]. \quad (7)$$

Note that, in the aforementioned structure, if $m = 0$ the designed filter is of *full-order* (i.e., the same order of the plant being filtered) and if $m > 0$ it has *non-minimal order*. See (Frezzatto et al., 2017) for details.

For design purposes, the following auxiliary system is defined
$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\xi)\,\tilde{x}(k) + \tilde{B}_w(\xi)\,w(k) \\ y(k) &= \tilde{C}_y(\xi)\,\tilde{x}(k) + \tilde{D}_y(\xi)\,w(k) \\ z(k) &= \tilde{C}_z(\xi)\,\tilde{x}(k) + \tilde{D}_z(\xi)\,w(k) \end{aligned} \quad (8)$$

where $\tilde{x} \in \mathbb{R}^{n+m}$, $m \in \mathbb{N}$,

$$\left[\begin{array}{c|c}\tilde{A}(\xi) & \tilde{B}(\xi) \\ \hline \tilde{C}_y(\xi) & \tilde{D}_y(\xi) \\ \tilde{C}_z(\xi) & \tilde{D}_z(\xi)\end{array}\right] = \left[\begin{array}{cc|c}\mathbb{A}_1 & \mathbb{A}_2 & \mathbb{B} \\ 0 & A(\xi) & B_w(\xi) \\ \hline 0 & C_y(\xi) & D_y(\xi) \\ 0 & C_z(\xi) & D_z(\xi)\end{array}\right] \tag{9}$$

with $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{B}$ being given constant matrices of suitable dimensions, and matrices (7) are redefined as

$$\left[\begin{array}{c}\mathsf{A}(\xi) \\ \hline \mathsf{C}(\xi)\end{array}\right] = \left[\begin{array}{cc}\tilde{A}(\xi) & 0 \\ B_f\tilde{C}_y(\xi) & A_f \\ \hline \tilde{C}_z(\xi) - D_f\tilde{C}_y(\xi) & -C_f\end{array}\right],$$
$$\left[\begin{array}{c}\mathsf{B}(\xi) \\ \hline \mathsf{D}(\xi)\end{array}\right] = \left[\begin{array}{c}\tilde{B}_w(\xi) \\ B_f\tilde{D}_{yw}(\xi) \\ \hline \tilde{D}_{zw}(\xi) - D_f\tilde{D}_{yw}(\xi)\end{array}\right]. \tag{10}$$

Matrices $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{B}$ are design parameters that, as discussed in Frezzatto et al. (2017), can be randomly chosen, provided that the pair $(\mathbb{A}_1, \mathbb{A}_2)$ is controllable. In this paper, it is assumed that these matrices have the following simple structure (that corresponds to the case of a memory filter as addressed in Frezzatto et al. (2017))

$$\mathbb{A}_1 = \begin{bmatrix} 0 & I & & \cdots & 0 \\ & 0 & \ddots & & \vdots \\ & & \ddots & \ddots & \\ \vdots & & & 0 & I \\ 0 & \cdots & & & 0 \end{bmatrix}, \quad \mathbb{A}_2 = \begin{bmatrix} 0 \\ \\ \vdots \\ 0 \\ I \end{bmatrix}, \quad \mathbb{B} = 0. \tag{11}$$

In brief, the main goal can be restated as: design a non-minimal filter as in (5) such that a bound to the $\mathscr{H}_\infty$ norm of the transfer function (6) with matrices given in (10) is minimized in a low-frequency range specification.

The next section presents sufficient parameter-dependent LMI conditions to achieve the aforementioned requirements.

4. MAIN RESULTS

The conditions in Lemma 2 already provide a suitable framework for low-frequency filtering design. However, they do not take full advantage of the system realization (10). To provide a sufficient condition for the robust non-minimal order low-frequency $\mathscr{H}_\infty$ filtering design problem, the following structures are adopted for the auxiliary matrices F, G and H:

$$F(\xi) = \left[\begin{array}{cc|c}\mathbb{X}_1(\xi) & X(\xi) & \hat{K} \\ \mathbb{Y}_1(\xi) & Y(\xi) & \hat{K}\end{array}\right], \qquad H = 0, \tag{12}$$

$$G(\xi) = \left[\begin{array}{cc|c}\mathbb{R}_1(\xi) & R(\xi) & 0 \\ \mathbb{S}_1(\xi) & S(\xi) & 0\end{array}\right]. \tag{13}$$

The last column block of matrix $F(\xi)$ has to be made independent from the uncertain parameter ξ in order to enable the synthesis of robust filters (parameter-independent) in the upcoming design condition. The last column block of $G(\xi)$ and the matrix H are zeroed for simplicity of presentation of the next theorem. Nevertheless, another possible choice would be to follow the lines of Lacerda et al. (2011, 2013), which impose some blocks of these variables to be equal (or proportional) to $\lambda\hat{K}$, $\lambda \in \mathbb{R}$. In this case, a line search on λ may improve the results at the price of a higher computational effort.

Theorem 1. Let $E_c = [I\ 0]$, $E = [0\ I]$, $\mathbb{A} = [\mathbb{A}_1\ \mathbb{A}_2]$, and

$$\left[\begin{array}{c}\mathscr{A}(\xi) \\ \hline \mathscr{C}(\xi)\end{array}\right] = \left[\begin{array}{cc}M_{Bf}C_y(\xi)E & M_{Af} \\ M_{Bf}C_y(\xi)E & M_{Af} \\ \hline C_z(\xi)E - D_fC_y(\xi)E & -C_f\end{array}\right],$$

$$\left[\begin{array}{c}\mathscr{B}(\xi) \\ \hline \mathscr{D}(\xi)\end{array}\right] = \left[\begin{array}{c}M_{Bf}D_y(\xi) \\ M_{Bf}D_y(\xi) \\ \hline D_z(\xi) - D_fD_y(\xi)\end{array}\right],$$

$$\mathscr{P}(\xi) = \begin{bmatrix}P_1(\xi) & P_2(\xi) \\ P_2(\xi)^T & P_3(\xi)\end{bmatrix}, \ \mathscr{Q}(\xi) = \begin{bmatrix}Q_1(\xi) & Q_2(\xi) \\ Q_2(\xi)^T & Q_3(\xi)\end{bmatrix},$$

$$\mathscr{G}(\xi) = \begin{bmatrix}0 & \hat{K} \\ 0 & \hat{K}\end{bmatrix},$$

$$\mathscr{X}(\xi) = \begin{bmatrix}X(\xi)^T & Y(\xi)^T & R(\xi)^T & S(\xi)^T & 0 & 0\end{bmatrix}^T,$$
$$\mathscr{R}(\xi) = [-E\ \ 0\ \ A(\xi)E\ \ 0\ \ B_w(\xi)\ \ 0],$$

$$\mathscr{S}(\xi) = \begin{bmatrix}-\mathscr{P}(\xi) - \mathscr{G}(\xi) - \mathscr{G}(\xi)^T & \star & \star & \star \\ \mathscr{Q}(\xi) + \mathscr{A}(\xi)^T & \mathscr{P}(\xi) - 2\cos(v_\ell)\mathscr{Q}(\xi) & \star & \star \\ \mathscr{B}(\xi)^T & 0 & -\gamma^2 I & \star \\ 0 & \mathscr{C}(\xi) & \mathscr{D}(\xi) & -I\end{bmatrix}.$$

For given scalars $\gamma > 0$ and $v_\ell \in [-\pi,\ \pi]$, $m \in \mathbb{N}$, if there exist matrices $P_1(\xi) = P_1(\xi)^T$, $P_3(\xi) = P_3(\xi)^T$, $P_2(\xi)$, $Q_1(\xi) = Q_1(\xi)^T$, $Q_3(\xi) = Q_3(\xi)^T$, $Q_2(\xi)$, $X(\xi)$, $Y(\xi)$, $R(\xi)$, $S(\xi)$, $\hat{K}$, M_{Af}, M_{Bf}, C_f, and D_f such that

$$\mathscr{Q}(\xi) \succ 0 \tag{14}$$

$$\mathscr{W}^{\perp^T}\left(\mathscr{S}(\xi) + \mathscr{X}(\xi)\mathscr{R}(\xi) + \mathscr{R}(\xi)^T\mathscr{X}(\xi)^T\right)\mathscr{W}^{\perp} \prec 0 \tag{15}$$

hold for all $\xi \in \Xi$, where

$$\mathscr{W}^{\perp} = \begin{bmatrix}E^T & 0 & E_c^T\mathbb{A} & 0 & E_c^T\mathbb{B} & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I\end{bmatrix}, \tag{16}$$

then there exists a filter realization $A_f = \hat{K}^{-1}M_{Af}$, $B_f = \hat{K}^{-1}M_{Bf}$, C_f, and D_f for which the connection of the non-minimal robust filter (5) with the uncertain system (8) is such that $\|H(\zeta,\xi)\|_\infty < \gamma$, for all $\zeta = e^{j\omega}$, $|\omega| < v_\ell$.

Proof. Assume that (14) and (15) are feasible and let

$$\mathscr{W} = [-E_c\ \ 0\ \ \mathbb{A}\ \ 0\ \ \mathbb{B}\ \ 0],$$

$$\mathbb{Z}(\xi) = \begin{bmatrix}\mathbb{X}_1(\xi)^T & \mathbb{Y}_1(\xi)^T & \mathbb{R}_1(\xi)^T & \mathbb{S}_1(\xi)^T & 0 & 0\end{bmatrix}^T.$$

Then, rewrite inequality (15) as in (17) (at the beginning of the next page). The $(2,2)$-block in inequality (17) implies that $\hat{K} + \hat{K}^T \succ -P_3(\xi)$. Due to matrix $P_3(\xi)$ being indefinite, it is possible in some cases that matrix $\hat{K}$ is singular or very close to singularity. In those cases, one can always choose an $\varepsilon > 0$ sufficiently small, such that $\hat{K} + \hat{K}^T \succ -P_3(\xi) + \varepsilon I$ (by homogeneity of the LMI condition) and assuring that conditions (14) and (15) are still feasible. Therefore, $\hat{K}$ can be assumed as a non-singular matrix. Thus, defining

$$\Upsilon = \begin{bmatrix}E_c^T\mathbb{A} + E^TA(\xi)E & 0 & E_c^T\mathbb{B} + E^TB_w(\xi) & 0 \\ \hat{K}^{-1}M_{Bf}C_y(\xi)E & \hat{K}^{-1}M_{Af} & \hat{K}^{-1}M_{Bf}D_y(\xi) & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I\end{bmatrix}$$

and multiplying (17) on the right by Υ and on the left by its transpose and noticing that $E_cE_c^T = I$, $EE^T = I$, and $E_cE^T = 0$,

$$0 \succ \mathscr{S}(\xi)+\mathscr{X}(\xi)\mathscr{R}(\xi)+\mathscr{R}(\xi)^T\mathscr{X}(\xi)^T+\mathbb{Z}(\xi)\mathscr{W}+\mathscr{W}^T\mathbb{Z}(\xi)^T =$$

$$\begin{bmatrix} -P_1(\xi)+\mathrm{He}\left(\mathbb{X}_1(\xi)E_c-X(\xi)E\right) & \star & \star & \star & \star & \star \\ -P_2(\xi)^T+\mathbb{Y}_1(\xi)E_c+Y(\xi)E-\hat{K}^T & -P_3(\xi)-\mathrm{He}\left(\hat{K}\right) & \star & \star & \star & \star \\ Q_1(\xi)-\mathbb{R}_1(\xi)E_c-R(\xi)E+\mathbb{A}^T\mathbb{X}_1(\xi)^T+E^TA(\xi)^TX(\xi)^T+E^TC_y^T(\xi)M_{Bf}^T & Q_2(\xi)+\mathbb{A}^T\mathbb{Y}_1(\xi)^T+E^TA(\xi)^TY(\xi)^T+E^TC_y^T(\xi)M_{Bf}^T & P_1(\xi)-2\cos(v_\ell)Q_1(\xi)+\mathrm{He}\left(\mathbb{R}_1(\xi)\mathbb{A}+R(\xi)A(\xi)E\right) & \star & \star & \star \\ Q_2(\xi)^T-\mathbb{S}_1(\xi)E_c-S(\xi)E+M_{Af}^T & Q_3(\xi)+M_{Af}^T & P_2(\xi)^T-2\cos(v_\ell)Q_2(\xi)^T+\mathbb{S}_1(\xi)\mathbb{A}+S(\xi)A(\xi)E & P_3(\xi)-2\cos(v_\ell)Q_3(\xi) & \star & \star \\ \mathbb{B}^T\mathbb{X}_1(\xi)^T+B_w(\xi)^TX(\xi)^T+D_y^T(\xi)M_{Bf}^T & \mathbb{B}^T\mathbb{Y}_1(\xi)^T+B_w(\xi)^TY(\xi)^T+D_y^T(\xi)M_{Bf}^T & \mathbb{B}^T\mathbb{R}_1(\xi)^T+B_w(\xi)^TR(\xi)^T & \mathbb{B}^T\mathbb{S}_1(\xi)^T+B_w(\xi)^TS(\xi)^T & -\gamma^2 I & \star \\ 0 & 0 & C_z(\xi)E-D_fC_y(\xi)E & -C_f(\xi) & D_z(\xi)-D_fD_y(\xi) & -I \end{bmatrix}. \tag{17}$$

$$\begin{bmatrix} \mathscr{P}(\xi)-\mathsf{A}(\xi)^T\mathscr{P}(\xi)\mathsf{A}(\xi)+\mathsf{A}(\xi)^T\mathscr{Q}(\xi)+\mathscr{Q}(\xi)\mathsf{A}(\xi)-2\cos(v_\ell)\mathscr{Q}(\xi) & \star & \star \\ \mathsf{B}(\xi)^T\mathscr{Q}(\xi)-\mathsf{B}(\xi)^T\mathscr{P}(\xi)\mathsf{A}(\xi) & -\mathsf{B}(\xi)^T\mathscr{P}(\xi)\mathsf{B}(\xi)-\gamma^2 I & \star \\ \mathsf{C}(\xi) & \mathsf{D}(\xi) & -I \end{bmatrix} \prec 0 \tag{18}$$

after some straightforward manipulations one gets (18), that is equal to (1) for uncertain systems by means of a Schur complement. Besides, matrices $\mathsf{A}(\xi)$, $\mathsf{B}(\xi)$, $\mathsf{C}(\xi)$, and $\mathsf{D}(\xi)$ are as in (7) after substituting the filter variables $A_f=\hat{K}^{-1}M_{Af}$ and $B_f=\hat{K}^{-1}M_{Bf}$.

The conditions of Theorem 1 may synthesize robust filters that are not asymptotically stable. Hence, a stability condition for the augmented system (10) must be added to the design conditions of Theorem 1. The next corollary provides conditions for the design of asymptotically stable non-minimal order low-frequency $\mathscr{H}_\infty$ filters.

Corollary 1. Let $E_c=[I\ 0]$, $E=[0\ I]$, $\mathbb{A}=[\mathbb{A}_1\ \mathbb{A}_2]$, $\mathscr{A}(\xi)$ and $\mathscr{G}(\xi)$ as in Theorem 1, and

$$\mathscr{M}_s(\xi)=\begin{bmatrix} M_1(\xi) & M_2(\xi) \\ M_2(\xi)^T & M_3(\xi)\end{bmatrix}, \tag{19}$$

$$\mathscr{X}_s(\xi)=\left[\varepsilon J(\xi)^T\ \ \varepsilon K(\xi)^T\ \ J(\xi)^T\ \ K(\xi)^T\right]^T, \tag{20}$$

$$\mathscr{B}_s(\xi)=[A(\xi)E\ \ 0\ \ E\ \ 0], \tag{21}$$

$$\mathscr{S}_s(\xi)= \begin{bmatrix} \mathscr{M}_s(\xi)+\varepsilon(\mathscr{A}(\xi)+\mathscr{A}(\xi)^T) & \star \\ \mathscr{A}(\xi)+\varepsilon\mathscr{G}(\xi)^T & \mathscr{G}(\xi)+\mathscr{G}(\xi)^T-\mathscr{M}_s(\xi)\end{bmatrix}. \tag{22}$$

For given scalars $\gamma>0$, $\varepsilon\in(-1,1)$, and $v_\ell\in[-\pi,\ \pi]$, $m\in\mathbb{N}$, if there exist matrices $P_1(\xi)=P_1(\xi)^T$, $P_3(\xi)=P_3(\xi)^T$, $P_2(\xi)$, $Q_1(\xi)=Q_1(\xi)^T$, $Q_3(\xi)=Q_3(\xi)^T$, $Q_2(\xi)$, $M_1(\xi)=M_1(\xi)^T$, $M_3(\xi)=M_3(\xi)^T$, $M_2(\xi)$, $X(\xi)$, $Y(\xi)$, $R(\xi)$, $S(\xi)$, $J(\xi)$, $K(\xi)$, $\hat{K}$, M_{Af}, M_{Bf}, C_f, and D_f such that (14), (15),

$$\mathscr{M}_s(\xi)\succ 0 \tag{23}$$

$$\mathscr{W}_s^{\perp T}\left(\mathscr{S}_s(\xi)+\mathscr{X}_s(\xi)\mathscr{B}_s(\xi)+\mathscr{B}_s(\xi)^T\mathscr{X}_s(\xi)^T\right)\mathscr{W}_s^{\perp}\succ 0 \tag{24}$$

hold for all $\xi\in\Xi_N$, where

$$\mathscr{W}_s^{\perp}=\begin{bmatrix} -I & 0 & 0 \\ 0 & -I & 0 \\ E_c^T\mathbb{A} & 0 & 0 \\ 0 & 0 & -I\end{bmatrix}, \tag{25}$$

then there exists a stable filter realization $A_f=\hat{K}^{-1}M_{Af}$, $B_f=\hat{K}^{-1}M_{Bf}$, C_f, and D_f for which the connection of the non-minimal robust filter (5) with the uncertain system (8) is such that $\|H(\zeta,\xi)\|_\infty<\gamma$, for all $\zeta=e^{j\omega}$, $|\omega|<v_\ell$.

Proof. Assume that (14), (15), (23), and (24) are feasible and let

$$\mathscr{W}_s=[\mathbb{A}\ \ 0\ \ E_c\ \ 0],$$

$$\mathbb{Z}_s(\xi)=\left[\varepsilon\mathbb{J}_1(\xi)^T\ \ \varepsilon\mathbb{K}_1(\xi)^T\ \ \mathbb{J}_1(\xi)^T\ \ \mathbb{K}_1(\xi)^T\right]^T.$$

Then, inequality (24) is rewritten as

$$\mathscr{S}_s(\xi)+\mathscr{X}_s(\xi)\mathscr{B}_s(\xi)+\mathscr{B}_s(\xi)^T\mathscr{X}_s(\xi)^T +\mathbb{Z}_S(\xi)\mathscr{W}_s+\mathscr{W}_s^T\mathbb{Z}_s(\xi)^T\prec 0. \tag{26}$$

Defining

$$\Upsilon_s=\begin{bmatrix} -I & 0 \\ 0 & -I \\ E_c^T\mathbb{A}+E^TA(\xi)E & 0 \\ \hat{K}^{-1}M_{Bf}C_y(\xi)E & \hat{K}^{-1}M_{Af}\end{bmatrix}$$

and multiplying (26) on the right by Υ_s and on the left by its transpose leads to

$$\mathscr{M}_s(\xi)-\mathsf{A}(\xi)^T\mathscr{M}_s(\xi)\mathsf{A}(\xi)\succ 0$$

after substituting the filter variables $A_f=\hat{K}^{-1}M_{Af}$ and $B_f=\hat{K}^{-1}M_{Bf}$.

Now, consider

$$\Psi=\begin{bmatrix} -I & 0 \\ 0 & -I \\ \varepsilon I & 0 \\ 0 & \varepsilon I\end{bmatrix}$$

and multiply (26) on the right by Ψ and on the left by Ψ^T yielding

$$\mathscr{M}_s(\xi) - \varepsilon^2 \mathscr{M}_s(\xi) \succ 0,$$

which assures that $|\varepsilon| < 1$.

The rest of the proof follows from the proof of Theorem 1.

5. NUMERICAL EXAMPLES

The examples presented in this section illustrate the performance of the proposed synthesis conditions. The routines were implemented in MATLAB, version 8.2.0.701 64 bits, using YALMIP (Löfberg, 2004) and Mosek (MOSEK ApS, 2015). To derive a finite set of LMIs that guarantees the feasibility of the proposed conditions, all parameter-dependent variables have been assumed with polynomial dependence of an arbitrary degree d. The LMIs were programmed with the aid of the package ROLMIP (Robust LMI Parser) (Agulhari et al., 2012).

Example 1. Consider the LTI system borrowed from Lee (2013)

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\delta \end{bmatrix} x(k) + \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix} w(k) \\ y(k) &= [-100 \;\; 10]\, x(k) + [0 \;\; 1]\, w(k) \\ z(k) &= [1 \;\; 0]\, x(k) \end{aligned} \tag{27}$$

where $|\delta| \leq 0.45$. Note that this system can be converted into a two-vertices polytopic representation. Table 1 shows the results for the application of Corollary 1 for $m = 0, 1, 2$ and 3, and $d = 1$. The number of scalar variables (V) and LMI rows (L) are also presented.

Table 1. Application of Corollary 1 for the system (27) in the frequency-range $|\omega| \leq \pi/6$. The first row is the result of Lee (2013).

	γ	V	L
Lee (2013)	1.18	–	–
$m = 0$	1.17	121	73
$m = 1$	1.10	215	93
$m = 2$	1.00	345	119
$m = 3$	1.00	503	145

This example illustrates the importance of the non-minimal filter structure, which yields a reduction of 14% in the value of the bound on the $\mathscr{H}_\infty$ norm. On the other hand, using the condition of Corollary 1 with $d = 6$ and $m = 0$ the value $\gamma = 1.17$ is obtained. This latter fact illustrates that the increase in the degree d of the decision variables, by itself, cannot guarantee that lower bounds will be obtained. For $d = 1$ and $m = 2$, the singular value diagram of the system (27) is depicted in Figure 1 for several values of $\xi \in \Xi$. The frequency range $|\omega| \leq \nu_\ell$ is shown in light blue and the bound $\gamma = 1.00$ is given by the red line. Note that the certificated value of $\gamma = 1.00$ furnishes indeed an upper bound for the $\mathscr{H}_\infty$ norm of this system in the frequency range considered, proving the efficiency of the proposed method. As a final remark on this example, no improvement on the bound was observed for m grater than 2.

Example 2. Consider the uncertain discrete-time system adapted from Wang and Yang (2008) given by

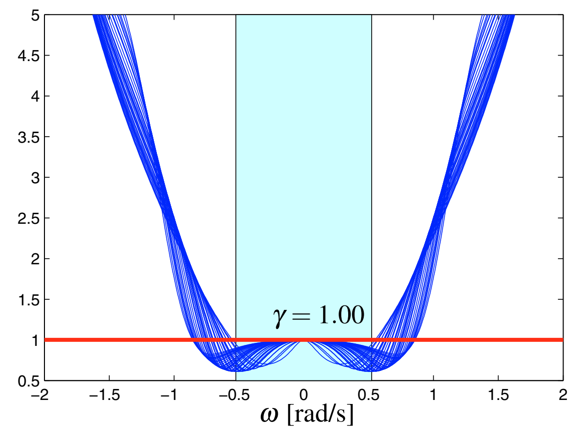

Fig. 1. Singular values diagram for system (27) connected with the filter designed by Corollary 1, $d = 1$, $m = 2$, $\nu_\ell = \pi/6$, for several values of $\xi \in \Xi$. The frequency range is given by the region in light blue and the bound provided by the proposed method, $\gamma = 1.00$, is shown by the red line.

$$\begin{aligned} x(k+1) &= \begin{bmatrix} -0.1996 & 0.1235+\rho \\ -1.8704 & -0.1457 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(k) \\ y(k) &= [1 \;\; 1]\, x(k) + w(k) \\ z(k) &= [1 \;\; 0]\, x(k) \end{aligned} \tag{28}$$

where $|\rho| \leq 0.3$. A polytopic representation is obtained by evaluating the uncertain parameter at the extreme points of the interval.

The aim is to analyze the behavior of the above system in the frequency range $|\omega| \leq 2\pi/3$. By applying the conditions of Corollary 1, robust filters are designed for several values of d and m, providing the results given in Table 2. The number of optimization variables V and LMI rows L involved in each problem is given as well. In this example, there is no improvement over the $\mathscr{H}_\infty$ norm bounds by varying the value of m if the degree of the parameter-dependent variables is $d = 0$, i.e., constant decision variables. On the other hand, letting the parameter-dependent variables to have an affine dependency on the uncertainty can reduce the certified upper-bounds and the increase of m provides small improvements over the $\mathscr{H}_\infty$ bounds.

Table 2. $\mathscr{H}_\infty$ bounds in the frequency range $|\omega| \leq 2\pi/3$ for the system (28) for several values of d and m.

d	m	γ	V	L
0	0	2.03	68	44
0	1	2.03	121	56
0	2	2.03	194	72
1	0	0.99	122	70
1	1	0.94	216	90
1	2	0.93	346	116
2	0	0.95	311	124
2	1	0.93	498	160
2	2	0.93	725	196

Considering the case $d = 1$ and $m = 2$, a diagram depicting the maximum singular values in the pre-specified frequency range is presented in Figure 2. The value of the worst-case $\mathscr{H}_\infty$ norm obtained from the diagram is 0.93 (computed by brute force), which corresponds to the certified upper-bound computed by Corollary 1. Contrast this with the worst-case norm 2.03 certified for the robust filter designed with $d = 0$ and $m = 2$. This shows that the proposed approach can provide improved performance for the system (7) and, moreover, reduce the gap between the determined upper-bound and the worst-case $\mathscr{H}_\infty$ norm.

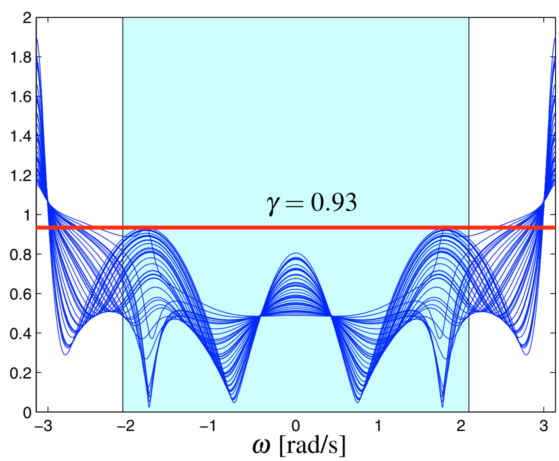

Fig. 2. Singular values diagram for system (28) connected with the filter designed by Corollary 1, $d = 1$, $m = 2$, for several values of $\xi \in \Xi$. The frequency range is given by the region in light blue and the bound provided by the proposed method, $\gamma = 0.93$, is shown by the red line.

6. CONCLUSION

This paper proposed a new approach to the robust $\mathscr{H}_\infty$ filtering design problem for uncertain linear discrete time-invariant systems using the gKYP lemma with low-frequency specifications. The approach relies on the use of non-minimal order filter structures, which combined with polynomial approximations for the decision variables of the parameter-dependent LMI conditions, provides improved $\mathscr{H}_\infty$ upper-bounds when compared with other methods from the literature that use standard full-order filter structures. Examples borrowed from the literature illustrate the effectiveness of the proposed approach. As future works, the authors aim to investigate the necessity part of the results as well as to address the problem of robust $\mathscr{H}_\infty$ filtering with middle and high frequency specifications.

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