$\mathscr{H}_\infty$ ROBUST FILTER DESIGN FOR CONTINUOUS-TIME LINEAR SYSTEMS USING LMIS WITH A SCALAR PARAMETER

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Abstract— This paper investigates the problem of $\mathscr{H}_\infty$ robust full order filter design for continuous-time uncertain systems in polytopic domains. A matrix inequality condition with a scalar is proposed for the filter design. The condition becomes a linear matrix inequality (LMI) for fixed values of the scalar parameter. The matrices of the filter are obtained through LMI relaxations that consider polynomially parameter-dependent decision variables of arbitrary degree. As main characteristic, the approach contains and generalizes the quadratic stability based LMI condition from the literature, providing the optimal $\mathscr{H}_\infty$ filter for precisely known systems. Additionally, numerical examples show that the proposed relaxations may require less computational effort in order to provide similar $\mathscr{H}_\infty$ bounds for uncertain systems when compared with other methods from the literature.

Keywords— Robust Filtering, LMIs, Continuous-Time Systems, Uncertain Systems, $\mathscr{H}_\infty$ Norm.

Resumo— Neste trabalho, investiga-se o problema de projeto de filtros robustos de ordem completa para sistemas lineares contínuos no tempo com incertezas politópicas tendo como critério de desempenho a norma $\mathscr{H}_\infty$. Uma condição na forma de desigualdade matricial com um parâmetro escalar é proposta para o projeto do filtro. A condição torna-se uma desigualdade matricial linear (LMI, de *Linear Matrix Inequality* em inglês) para valores fixos do escalar. As matrizes do filtro são obtidas por meio de relaxações LMIs considerando variáveis de decisão polinomialmente dependentes de parâmetro com grau arbitrário . Como característica principal, a abordagem contém e generaliza as condições LMIs baseadas na estabilidade quadrática da literatura, provendo o filtro ótimo $\mathscr{H}_\infty$ para sistemas precisamente conhecidos. Adicionalmente, exemplos numéricos mostram que as relaxações propostas podem exigir esforço computacional menor para produzir limitantes $\mathscr{H}_\infty$ similares quando comparadas a outros métodos da literatura.

Palavras-chave— Filtragem Robusta, LMIs, Sistemas Contínuos no Tempo, Sistemas Incertos, Norma $\mathscr{H}_\infty$.

1 Introduction

Over the last decades, the modeling of dynamical systems in several areas of control engineering has incorporated more complex assumptions on physical components, for instance, taking into account the presence of uncertainties. To cope with more sophisticated models, one of the main strategies relies on the Lyapunov stability theory, that has been employed to solve a wide range of problems in terms of convex optimization based on Linear Matrix Inequalities (LMIs) (Boyd et al., 1994).

The LMI framework was first used in the context of robust stability analysis, exploring the concept of quadratic stability, i.e., a constant Lyapunov matrix is used to certify the stability of the whole domain of uncertainty. Then, extensions were provided to deal with the controller and filtering design problems. Although quadratic stability provides optimal full order controllers and filters in the sense of some performance criteria, such as the $\mathscr{H}_2$ and the $\mathscr{H}_\infty$ norms, for precisely known systems, the results may be conservative in the uncertain case. As a consequence, robust stability analysis conditions based on affine parameter-dependent Lyapunov functions (de Oliveira et al., 1999; Leite and Peres, 2003) have been employed to reduce the conservativeness and lately, polynomial parameter-dependent Lyapunov functions of arbitrary degree further improved the results by means of convergent LMI relaxations (Bliman, 2004; Oliveira et al., 2008).

The filtering problem is an important topic in signal processing and control theory that has been investigated for a long time. Although the Kalman filter theory provides the optimal unbiased estimate of the unknown state for the discrete-time system in terms of minimum variance error, if one considers the presence of uncertainties in the model, the filter can no longer assure such optimality property. In addition, the Kalman filter theory has limitations regarding the noise input, which is required to be Gaussian.

Due to the aforementioned reasons, control engineers have been focused on developing extensions and generalizations for the Kalman filter. Therefore, alternative ways to deal with the filtering problem were proposed and one of the most popular approaches is the LMI based technique, that has provided synthesis conditions for filter design based on quadratic stability in the continuous-time case (Geromel and de Oliveira, 2001; Geromel, 1999; de Souza et al., 2000), discrete-time case (Geromel et al., 2000) and systems with delays (Zhong, 2006), generally using the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ norms of the transfer matrix from the input noise to the estimated output error as performance criteria.

Recently, filter design conditions which are more general and can provide less conservative results have been proposed (Duan et al., 2006; Gao et al., 2008; Lacerda et al., 2011). Those conditions make use of extra degrees of freedom provided by scalar parameters, that can be useful to provide less conservative results, at the expense of increasing the computational burden.

This paper proposes a new LMI based design condition for $\mathscr{H}_\infty$ filtering of continuous-time linear systems, which can be viewed as an extension of the LMIs with a scalar parameter proposed in (Morais et al., 2013) for the problem of robust stabilization. The approach contains and generalizes the quadratic stability based conditions for filter design, always yielding less conservative $\mathscr{H}_\infty$ guaranteed costs and providing the optimal $\mathscr{H}_\infty$ filter when precisely known systems are investigated. Moreover, the conditions may require less computational effort to provide similar bounds when compared to other available conditions for robust filtering. The matrices of the dynamic filter are obtained in terms of some blocks of the slack variables of the problem, allowing both the Lyapunov matrix and other blocks of the slack variables to be polynomially parameter-dependent. This approach can be useful to further reduce the conservativeness of the proposed condition in the case of uncertain systems. Numerical examples illustrate the results.

2 Preliminaries

Consider the robustly stable uncertain linear continuous time-invariant system described by

$$\begin{aligned} \dot{x}(t) &= A(\alpha)x(t) + B_1(\alpha)w(t) \\ z(t) &= C_1(\alpha)x(t) + D_{11}(\alpha)w(t) \\ y(t) &= C_2(\alpha)x(t) + D_{21}(\alpha)w(t) \end{aligned} \tag{1}$$

with $A(\alpha) \in \mathbb{R}^{n\times n}$, $B_1(\alpha) \in \mathbb{R}^{n\times r}$, $C_1(\alpha) \in \mathbb{R}^{p\times n}$, $D_{11}(\alpha) \in \mathbb{R}^{p\times r}$, $C_2(\alpha) \in \mathbb{R}^{q\times n}$, $D_{21}(\alpha) \in \mathbb{R}^{q\times r}$, where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^r$ is the noise input, $z \in \mathbb{R}^p$ is the output to be estimated and $y \in \mathbb{R}^q$ represents the measured output. The matrices of the system are supposed to belong to a polytope defined by

$$\mathscr{X} = \left\{R(\alpha) : R(\alpha) = \sum_{i=1}^{N} \alpha_i R_i\right\},$$

where $R(\alpha)$ represents any matrix of system (1), R_i are given matrices (vertices of the polytope), N is the number of vertices and the parameter α lies in the unit simplex of dimension N, given by

$$\Lambda_N = \left\{\alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1,\dots,N\right\}.$$

The purpose is the design of a full order stable filter with the following state-space representation

$$\begin{aligned} \dot{x}_f(t) &= A_f x_f(t) + B_f y(t) \\ z_f(t) &= C_f x_f(t) + D_f y(t), \end{aligned} \tag{2}$$

where $x_f \in \mathbb{R}^{n_f}$, with $n_f = n$, and $z_f \in \mathbb{R}^p$ is the filter output, in order to minimize a bound to the $\mathscr{H}_\infty$ norm of the transfer matrix from the noise input, w, to the estimation error, $e = z - z_f$.

Defining an augmented system by merging the states of system (1) and the filter (2) into a single state vector $\tilde{x}^T = \begin{bmatrix} x^T & x_f^T \end{bmatrix}$, the problem can be stated in terms of the augmented system

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}(\alpha)\tilde{x}(t) + \tilde{B}(\alpha)w(t) \\ e(t) &= \tilde{C}(\alpha)\tilde{x}(t) + \tilde{D}(\alpha)w(t), \end{aligned} \tag{3}$$

where

$$\tilde{A}(\alpha) = \begin{bmatrix} A(\alpha) & 0 \\ B_f C_2(\alpha) & A_f \end{bmatrix}, \quad \tilde{B}(\alpha) = \begin{bmatrix} B_1(\alpha) \\ B_f D_{21}(\alpha) \end{bmatrix},$$

$$\tilde{C}(\alpha) = \begin{bmatrix} C_1(\alpha) - D_f C_2(\alpha) & -C_f \end{bmatrix},$$

$$\tilde{D}(\alpha) = \begin{bmatrix} D_{11}(\alpha) - D_f D_{21}(\alpha) \end{bmatrix},$$

and the transfer matrix from w to e is given by

$$H(s,\alpha) = \tilde{C}(\alpha)\left(sI - \tilde{A}(\alpha)\right)^{-1}\tilde{B}(\alpha) + \tilde{D}(\alpha), \tag{4}$$

where s denotes the Laplace variable. Then, the $\mathscr{H}_\infty$ robust filtering design problem can be viewed as the minimization of a bound to the $\mathscr{H}_\infty$ norm of the transfer matrix $H(s,\alpha)$, $\forall \alpha \in \Lambda_N$.

Hence, this paper focuses on obtaining the filter matrices A_f, B_f, C_f and D_f by means of LMI conditions with a scalar parameter such that an upper bound for the $\mathscr{H}_\infty$ worst case norm of the transfer function (4) is minimized, for all $\alpha \in \Lambda_N$.

The $\mathscr{H}_\infty$ guaranteed-cost of system (3) can be characterized by the so-called Bounded Real Lemma (Boyd et al., 1994).

Lemma 1 *The inequality $\|H(s,\alpha)\|_\infty < \gamma$ holds for all $\alpha \in \Lambda_N$ if and only if there exists a parameter-dependent positive symmetric matrix $P(\alpha) \in \mathbb{R}^{2n\times 2n}$ such that*[1]

$$\begin{bmatrix} \tilde{A}(\alpha)^T P(\alpha) + P(\alpha)\tilde{A}(\alpha) & P(\alpha)\tilde{B}(\alpha) & \tilde{C}(\alpha)^T \\ \star & -I & \tilde{D}(\alpha)^T \\ \star & \star & -\gamma^2 I \end{bmatrix} \prec 0. \tag{5}$$

An alternative condition, including slack variables and a scalar parameter, to calculate a bound for the $\mathscr{H}_\infty$ norm of the system (3) is presented in the following lemma.

Lemma 2 *The inequality $\|H(s,\alpha)\|_\infty < \gamma$ holds for all $\alpha \in \Lambda_N$ if there exist a parameter-dependent symmetric positive definite matrix $P(\alpha) \in \mathbb{R}^{2n\times 2n}$, a parameter-dependent matrix $X(\alpha) \in \mathbb{R}^{2n\times 2n}$ and a scalar parameter $\xi > 0$ such that*

$$\begin{bmatrix} \mathscr{P}_{11} & \mathscr{P}_{12} & X(\alpha)^T\tilde{B}(\alpha) & \tilde{C}(\alpha)^T \\ \star & \mathscr{P}_{22} & \xi X(\alpha)^T\tilde{B}(\alpha) & 0 \\ \star & \star & -I & \tilde{D}(\alpha)^T \\ \star & \star & \star & -\gamma^2 I \end{bmatrix} \prec 0, \tag{6}$$

$$\begin{aligned} \mathscr{P}_{11} &= \tilde{A}(\alpha)^T X(\alpha) + X(\alpha)^T \tilde{A}(\alpha), \\ \mathscr{P}_{12} &= P(\alpha) + \xi\tilde{A}(\alpha)^T X(\alpha) - X(\alpha)^T, \\ \mathscr{P}_{22} &= -\xi(X(\alpha) + X(\alpha)^T). \end{aligned}$$

[1]The symbol $\star$ represents a symmetric block.

Proof: To prove the result, suppose that (6) holds with $\xi > 0$ and note that inequality (6) can be written as

$$\mathscr{Q} + \underbrace{\begin{bmatrix} \tilde{A}(\alpha)^T \\ -I \\ \tilde{B}(\alpha)^T \\ 0 \end{bmatrix} X(\alpha) \begin{bmatrix} I & \xi I & 0 & 0 \end{bmatrix}}_{\Psi} + \Psi^T \prec 0, \quad (7)$$

with

$$\mathscr{Q} = \begin{bmatrix} 0 & P(\alpha) & 0 & \tilde{C}(\alpha)^T \\ P(\alpha) & 0 & 0 & 0 \\ 0 & 0 & -I & \tilde{D}(\alpha)^T \\ \tilde{C}(\alpha) & 0 & \tilde{D}(\alpha) & -\gamma^2 I \end{bmatrix}.$$

Multiplying (7) on the right by

$$\Xi(\alpha) = \begin{bmatrix} I & 0 & 0 \\ A(\alpha) & B(\alpha) & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and on the left by $\Xi(\alpha)^T$ yields (5). Then, the result follows from Lemma 1. □

Notice that, for a given robust filter realization (A_f, B_f, C_f, D_f), the conditions of both lemmas 1 and 2 (for a fixed value of $\xi > 0$) are parameter-dependent LMIs that can be solved through LMI relaxations, providing bounds to the worst case $\mathscr{H}_\infty$ norm of the transfer matrix (4) for all $\alpha \in \Lambda_N$.

3 Main Result

Theorem 1 *The inequality $\|H(s,\alpha)\|_\infty < \gamma$ holds for all $\alpha \in \Lambda_N$ if there exist matrices $J(\alpha) \in \mathbb{R}^{n\times n}$, $X_0(\alpha) \in \mathbb{R}^{n\times n}$, $X_1(\alpha) \in \mathbb{R}^{n\times n}$, $X_5 \in \mathbb{R}^{n\times n}$, $M_{A_f} \in \mathbb{R}^{n\times n}$, $M_{B_f} \in \mathbb{R}^{n\times q}$, $M_{C_f} \in \mathbb{R}^{p\times n}$, $D_f \in \mathbb{R}^{p\times q}$, symmetric matrices $0 \prec E(\alpha) \in \mathbb{R}^{n\times n}$, $0 \prec G(\alpha) \in \mathbb{R}^{n\times n}$ and a scalar $\xi > 0$ such that*

$$\begin{bmatrix} E(\alpha) & J(\alpha) \\ \star & G(\alpha) \end{bmatrix} \succ 0, \quad (8)$$

and (9) *(top of the next page), hold for all $\alpha \in \Lambda_N$ with*

$$S_{15} = X_1(\alpha)B_1(\alpha) + M_{B_f}D_{21}(\alpha),$$
$$S_{25} = X_0(\alpha)B_1(\alpha) + M_{B_f}D_{21}(\alpha).$$

Then,

$$A_f = X_3^{-1}(M_{A_f}X_5^{-1})X_3, \quad B_f = X_3^{-1}M_{B_f}, \quad C_f = (M_{C_f}X_5^{-1})X_3 \quad (10)$$

and D_f, are the matrices of the filter assuring that γ is an $\mathscr{H}_\infty$ guaranteed-cost for system (3).

Proof: Notice that (9) can be written as[2]

$$\left[\begin{array}{c:c:c:c} \tilde{S}_{11} & \tilde{S}_{12} & \tilde{S}_{13} & \tilde{S}_{14} \\ \hdashline \star & \tilde{S}_{22} & \xi\tilde{S}_{13} & 0 \\ \hdashline \star & \star & -I & D_{11}^T - D_{21}^T D_f^T \\ \hdashline \star & \star & \star & -\gamma^2 I \end{array}\right] \prec 0 \quad (11)$$

with[3]

$$\tilde{S}_{11} = \begin{bmatrix} He(X_1A + M_{B_f}C_2) & M_{A_f} + A^TX_0^T + C_2^TM_{B_f}^T \\ \star & He(M_{A_f}) \end{bmatrix},$$

$$\tilde{S}_{12} = \begin{bmatrix} E + \xi(A^TX_1^T + C_2^TM_{B_f}^T) - X_1 & \mathscr{L}_{12} \\ J^T + \xi M_{A_f}^T - X_0 & G + \xi M_{A_f}^T - X_5 \end{bmatrix},$$

$$\tilde{S}_{13} = \begin{bmatrix} X_1B_1 + M_{B_f}D_{21} \\ X_0B_1 + M_{B_f}D_{21} \end{bmatrix}, \quad \tilde{S}_{14} = \begin{bmatrix} C_1^T - C_2^TD_f^T \\ -M_{C_f}^T \end{bmatrix},$$

$$\tilde{S}_{22} = \begin{bmatrix} -\xi He(X_1) & -\xi(X_5 + X_0^T) \\ \star & -\xi He(X_5) \end{bmatrix},$$

and $\mathscr{L}_{12} = J + \xi(A^TX_0^T + C_2^TM_{B_f}^T) - X_5$. Additionally, notice that an alternative way to write the above blocks

$$\tilde{S}_{11} = \tilde{T}^T(\tilde{A}^TX + X^T\tilde{A})\tilde{T}, \quad \tilde{S}_{12} = \tilde{T}^T(P + \xi\tilde{A}^TX - X^T)\tilde{T},$$
$$\tilde{S}_{13} = \tilde{T}^TX^T\tilde{B}, \quad \tilde{S}_{14} = \tilde{T}^T\tilde{C}, \quad \tilde{S}_{22} = \tilde{T}^T(X + X^T)\tilde{T}$$

where

$$X^T = \begin{bmatrix} X_1 & X_3 \\ X_4 & X_2 \end{bmatrix}, \quad \tilde{T} = \begin{bmatrix} I & 0 \\ 0 & X_2^{-T}X_3^T \end{bmatrix},$$

$$M_{A_f} = X_3A_fX_2^{-T}X_3^T, \quad X_0 = X_3X_2^{-1}X_4,$$

$$M_{B_f} = X_3B_f, \quad M_{C_f} = C_fX_2^{-T}X_3^T, \quad X_5 = X_3X_2^{-T}X_3^T.$$

and that variables X_1 and X_4 can be chosen to parameter-dependent, while variables X_3 and X_2 must be constant, since they are used to construct the matrices of the filter.

Consequently, (9) can be rewritten as

$$Z_1^{-T}\underbrace{\begin{bmatrix} \mathscr{P}_{11} & \mathscr{P}_{12} & X^T\tilde{B} & \tilde{C}^T \\ \star & \mathscr{P}_{22} & \xi X^T\tilde{B} & 0 \\ \star & \star & -I & \tilde{D}^T \\ \star & \star & \star & -\gamma^2 I \end{bmatrix}}_{\text{left-hand side of (6)}} \underbrace{\text{diag}\{\tilde{T}, \tilde{T}, I, I\}}_{Z_1^{-1}} \prec 0$$

As a result, by multiplying (11) by Z_1^T on the left and by Z_1 on the right, condition (6) is obtained. Finally, since $\tilde{T}^TP\tilde{T}$ produces the left-hand side of (8) and, by a congruence argument, one can conclude that $P(\alpha) \succ 0$. Hence, the result is proved invoking Lemma 2. □

The matrices of the filter that ensures an $\mathscr{H}_\infty$ performance for system (3) bounded by γ can be obtained by means of robust LMIs (for a fixed value of ξ) through the condition of Theorem 1. As an important remark, note that in the condition of Theorem 1, not only the Lyapunov matrix can be chosen to be polynomially-dependent

[2] For simplicity, the dependence in α is omitted.

[3] For any matrix A, $\text{He}(A) = A + A^T$, and A^{-T} stands for the transpose of the inverse of A.

$$
\begin{bmatrix}
\begin{pmatrix} He(X_1(\alpha)A(\alpha) \\ +M_{B_f}C_2(\alpha)) \end{pmatrix} & \begin{pmatrix} M_{A_f}+A(\alpha)^T X_0(\alpha)^T \\ +C_2(\alpha)^T M_{B_f}^T \end{pmatrix} & \begin{pmatrix} E(\alpha)+\xi(A(\alpha)^T X_1(\alpha)^T \\ +C_2(\alpha)^T M_{B_f}^T)-X_1(\alpha) \end{pmatrix} & \begin{pmatrix} J(\alpha)+\xi(A(\alpha)^T X_0(\alpha)^T \\ +C_2(\alpha)^T M_{B_f}^T)-X_5 \end{pmatrix} & S_{15} & \begin{pmatrix} C_1(\alpha)^T \\ -C_2(\alpha)^T D_f^T \end{pmatrix} \\
\star & He(M_{A_f}) & J(\alpha)^T+\xi M_{A_f}^T - X_0(\alpha) & G(\alpha)+\xi M_{A_f}^T - X_5 & S_{25} & -M_{C_f}^T \\
\star & \star & -\xi He(X_1(\alpha)) & -\xi(X_5+X_0(\alpha)^T) & \xi S_{15} & 0 \\
\star & \star & \star & -\xi He(X_5) & \xi S_{25} & 0 \\
\star & \star & \star & \star & -I & \begin{pmatrix} D_{11}(\alpha)^T \\ -D_{21}(\alpha)^T D_f^T \end{pmatrix} \\
\star & \star & \star & \star & \star & -\gamma^2 I
\end{bmatrix} \prec 0 \qquad (9)
$$

on the uncertainty but also the first column of the slack variable X^T, potentially leading to less conservative $\mathscr{H}_\infty$ guaranteed-costs. In addition, the bound provided by solving (9) depends on the value of $\xi > 0$. Therefore, a search on ξ is of crucial importance to further reduce the $\mathscr{H}_\infty$ guaranteed-cost in the uncertain case.

Lemma 3 *Theorem 1 contains the $\mathscr{H}_\infty$ filter design condition proposed in (Geromel and de Oliveira, 2001) (quadratic stability) when $\xi \to 0^+$.*

Proof: After pre and post-multiplying (11) by Z_1^T and Z_1, respectively, and defining $X = X^T = P$ (independent of α), and interchanging some rows and columns, (9) can be rewritten as

$$
\begin{bmatrix}
\tilde{A}(\alpha)^T P + P\tilde{A}(\alpha) & P\tilde{B}(\alpha) & \tilde{C}(\alpha)^T & \xi \tilde{A}(\alpha)^T P \\
\star & -I & \tilde{D}(\alpha)^T & \xi \tilde{B}(\alpha)^T P \\
\star & \star & -\gamma^2 I & 0 \\
\star & \star & \star & -2\xi P
\end{bmatrix} \prec 0. \qquad (12)
$$

Applying Schur's complement, the following equivalent condition is obtained

$$
\begin{bmatrix}
\tilde{A}(\alpha)^T P + P\tilde{A}(\alpha) & P\tilde{B}(\alpha) & \tilde{C}(\alpha)^T \\
\star & -I & \tilde{D}(\alpha)^T \\
\star & \star & -\gamma^2 I
\end{bmatrix}
+ \frac{\xi}{2}
\begin{bmatrix}
\tilde{A}(\alpha)^T P\tilde{A}(\alpha) & \tilde{A}(\alpha)^T P\tilde{B}(\alpha) & 0 \\
\star & \tilde{B}(\alpha)^T P\tilde{B}(\alpha) & 0 \\
\star & \star & 0
\end{bmatrix} \prec 0 \qquad (13)
$$

Notice that, by definition, matrix P is partitioned similarly to matrix X. The condition in (Geromel and de Oliveira, 2001) can be recovered by making $\xi \to 0^+$ in Theorem 1. □

In other words, Lemma 3 assures that the condition of Theorem 1 always provides the $\mathscr{H}_\infty$ optimal filter in the case of precisely known systems.

4 Numerical Experiments

This section aims to compare the performance of the proposed condition with other conditions available in the literature, thereby exposing the importance of the scalar parameter. The routines were implemented in MATLAB, version 8.2.0.701 64 bits, using Yalmip (Löfberg, 2004), SeDuMi (Sturm, 1999) and a PC with Ubuntu 14.04 LTS running as OS.

Example 1: Consider the randomly generated uncertain continuous-time system given by

$$
\left[\begin{array}{c|c} A_1 & B_{11} \\ \hline A_2 & B_{12} \end{array}\right] =
\left[\begin{array}{ccc|c}
-2.28 & 0.39 & -1.21 & 0.02 \\
-1.16 & -1.17 & 0.83 & -1.09 \\
0.58 & -0.62 & -0.99 & -1.12 \\ \hline
0.22 & -1.54 & 0.63 & -1.94 \\
1.19 & -1.73 & 0.39 & -0.38 \\
1.43 & -0.64 & -2.71 & -0.81
\end{array}\right],
$$

$$
\left[\begin{array}{c|c} C_{11}^T & C_{12}^T \\ \hline D_{111} & D_{112} \end{array}\right] =
\left[\begin{array}{c|c}
-0.49 & -0.63 \\
0.56 & 2.22 \\
-0.90 & -0.07 \\ \hline
0.76 & 0.37
\end{array}\right], \qquad (14)
$$

$$
\left[\begin{array}{c|c} C_{21}^T & C_{22}^T \\ \hline D_{211} & D_{212} \end{array}\right] =
\left[\begin{array}{c|c}
-0.49 & -0.63 \\
0.56 & 2.22 \\
-0.90 & -0.07 \\ \hline
-1.37 & 0.13
\end{array}\right].
$$

Figure 1 shows the influence of the scalar ξ on obtaining an $\mathscr{H}_\infty$ guaranteed-cost and a robust filter for this system. In this example, a search procedure in the scalar parameter yields a less conservative upper bound for the $\mathscr{H}_\infty$ guaranteed-cost in the presence of uncertainty 57% smaller than the one obtained with quadratic stability. Furthermore, defining $E(\alpha) = E$, $G(\alpha) = G$, $J(\alpha) = J$, $X_4(\alpha) = X_3^T = J^T$, $X_1(\alpha) = E$ and $X_2 = G$, the condition of Theorem 1 provides the filter obtained with the quadratic condition for sufficiently small value of ξ, as expected.

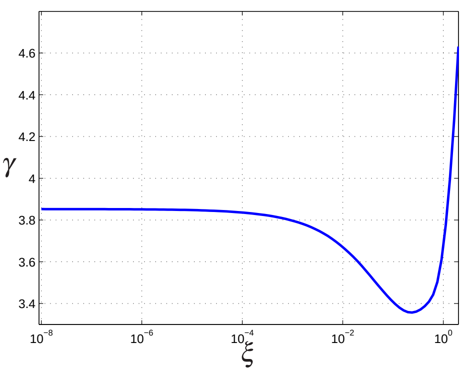

Figure 1: Effect of the scalar parameter ξ on obtaining an $\mathscr{H}_\infty$ guaranteed-cost for the uncertain system of Example 1.

In order to deeply explore the results of Example 1, consider the state-space realization of the filter provided by the quadratic stability, for

$\gamma = 7.90$,

$$\left[\begin{array}{c:c} A_f & B_f \\ \hdashline C_f & D_f \end{array}\right] = \left[\begin{array}{ccc:c} -1.60 & -1.31 & -1.21 & -2.27 \\ -5.38 & -1.68 & 3.76 & -2.22 \\ 5.67 & -0.60 & -4.78 & -2.68 \\ \hdashline -1.37 & -1.45 & -0.74 & -0.43 \end{array}\right], \quad (15)$$

and by the condition of the Theorem 1 with $\xi = 0.24$, the value that provides the less conservative result, $\gamma = 3.36$ (see Figure 1),

$$\left[\begin{array}{c:c} A_f & B_f \\ \hdashline C_f & D_f \end{array}\right] = \left[\begin{array}{ccc:c} -4.39 & -1.09 & -0.81 & 0.19 \\ -37.15 & -10.27 & -3.43 & -1.66 \\ 25.51 & 5.22 & 1.17 & 0.89 \\ \hdashline -59.27 & -19.79 & -10.51 & 0.64 \end{array}\right]. \quad (16)$$

With filters (15), (16) and the matrices given in (14), system (3) is completely defined. Figure 2 shows the singular value diagrams for 101 equally-spaced $\alpha \in \Lambda_2$ and the bounds provided by the quadratic approach (top) and Theorem 1 (bottom). Note that the bound provided by the condition of Theorem 1 is less conservative than the bound provided by the quadratic stability (i.e., closer to the actual case norm).

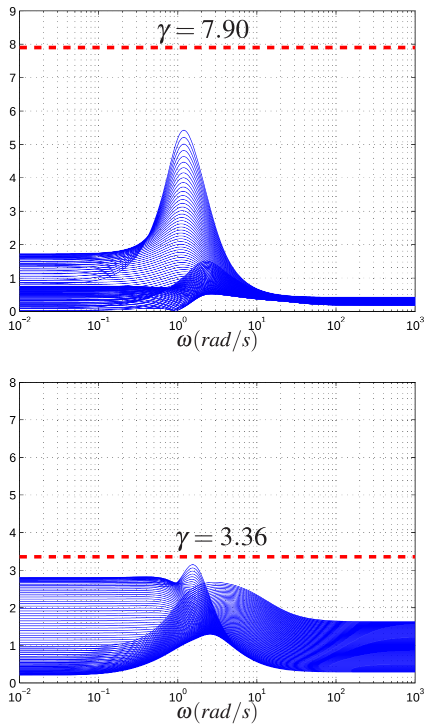

Figure 2: Singular value diagrams for system (3) with filter (15) (quadratic stability, top) or (16) (Theorem 1, bottom).

Example 2: In order to compare the condition of Theorem 1 with the results presented in (Geromel and de Oliveira, 2001) and (Lacerda et al., 2011), a numerical experiment[4] that generates a system with gradually increasing uncertainty has been implemented. To do so, a precisely known system with matrices A, B_1, C_1, D_{11}, C_2, D_{21} was generated. Then, an uncertain system with two vertices was created by the following procedure: $A_1 = A$, $A_2 = A - \rho I$, while the other matrices are kept constant. The uncertainty is iteratively increased from 0 to 0.5 with displacements of 0.05. Notice that for $\rho = 0$, the system is in fact precisely known. The matrices (randomly generated) used in this example are

$$A = \begin{bmatrix} -1.23 & -0.63 & -2.49 \\ 0.19 & -0.17 & 0.20 \\ 0.35 & 2.35 & -0.31 \end{bmatrix}, \; B_1 = \begin{bmatrix} -2.39 & -1.80 \\ 0.27 & 2.12 \\ 0.51 & 3.64 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.67 & -1.64 & -0.27 \\ 1.11 & 3.36 & 1.32 \end{bmatrix}, \; D_{11} = \begin{bmatrix} 1.17 & -0.19 \\ -0.30 & 0.09 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -0.67 & -3.81 & 1.45 \end{bmatrix}, \; D_{21} = \begin{bmatrix} -4.32 & -0.91 \end{bmatrix}.$$

Figure 3 illustrates the results. The condition of Theorem 1 provides a competitive $\mathcal{H}_\infty$ guaranteed-cost when compared to (Lacerda et al., 2011) in the specified range of uncertainty using less variables (108 versus 168). In this example, for both Theorem 1 and (Lacerda et al., 2011) ($\lambda_1 = \lambda_2 = \xi, \lambda_3 = \lambda_4 = 0$) a search for the values of ξ lying in the set

$$\Delta = \{\xi \in \mathbb{R} : \xi = 10^i, \quad i = -6, -5, \ldots, 5, 6\}$$

has been used. Note that the bound provided by Theorem 1 is always less conservative than the quadratic stability bound from (Geromel and de Oliveira, 2001) and remains close to the bound obtained with the conditions in (Lacerda et al., 2011) for the amount of uncertainty considered. For larger values of ρ, the method (Lacerda et al., 2011) provides clearly less conservative bounds.

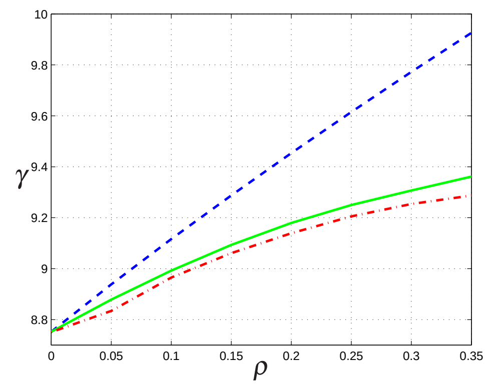

Figure 3: Bounds for the $\mathcal{H}_\infty$ guaranteed-cost for the system of Example 2 for (Lacerda et al., 2011) (dashed-dot red line), (Geromel and de Oliveira, 2001) (dashed blue line) and Theorem 1 (green line).

Example 3: Consider the following system borrowed from (Lacerda et al., 2011)

$$A = \begin{bmatrix} -0.6 & 4+\delta \\ -4 & -0.6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 1.5 & 0 \end{bmatrix}$$

$$C_1 = D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & -1.2 \end{bmatrix},$$

[4] In this example, the variables of the conditions in (Lacerda et al., 2011) and Theorem 1 were chosen to be affine-dependent in α wherever possible.

where $|\delta| \leq \bar{\delta}$. Table 1 shows the $\mathscr{H}_\infty$ guaranteed-costs obtained with Theorem 1 ($\xi = 0.1$) and those of (Lacerda et al., 2011, $\lambda_1 = \lambda_2 = 1$), for several values of $\bar{\delta}$. The importance of the results relies on the fact that the proposed condition may provide a competitive upper bound for the worst case $\mathscr{H}_\infty$ norm when compared to (Lacerda et al., 2011) while using less variables (50 versus 78) and the same number of LMI rows ($= 41$).

Table 1: $\mathscr{H}_\infty$ costs for the Example 3 using Theorem 1 (T1), $\xi = 0.1$, and (Lacerda et al., 2011) (LOP) with $\lambda_1 = \lambda_2 = 1$ for several values of $\bar{\delta}$.

$\bar{\delta}$	1	1.3	1.5	2	2.5
LOP	0.702	0.706	0.709	0.721	0.737
(T1)	0.702	0.707	0.711	0.724	0.741

5 Conclusions

A matrix inequality with a scalar parameter was proposed to cope with the $\mathscr{H}_\infty$ robust filtering design problem for continuous-time uncertain systems. Comparisons with other well-known results in the literature have been made in order to evaluate the effectiveness of the proposed method. The proposed condition becomes an LMI for fixed values of the scalar parameter and contains the quadratic stability result as a particular case. Moreover, the robust filters designed through the proposed approach provide competitive bounds for the worst case $\mathscr{H}_\infty$ norm when compared to other results using less variables.

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