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State-Feedback and Filtering Problems using the Generalized KYP Lemma

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Abstract— **The Kalman-Yakubovich-Popov (KYP) lemma is a useful tool in control theory, since it relates frequency-domain inequalities (FDIs) to linear matrix inequalities (LMIs). In the last decade, the standard KYP lemma was extended to cope with finite frequency intervals, namely low, middle and high frequencies. This extension is known as the generalized KYP lemma (gKYP). In this paper, a necessary and sufficient condition to assess middle-frequency specifications with an additional multiplier is proposed. This result is then used to address three problems in control theory: state-feedback control, observer-based estimation, and filter design. The drawback of the technique is that the extra multiplier must be complex, yielding complex matrices for the state feedback and observer-based estimation gains, as well as for the matrices of the filter realization. By imposing the multiplier to be real, sufficient design conditions are obtained, whose conservativeness is analyzed through numerical examples.**

I. Introduction

One of the most important results in control theory is the Kalman-Yakubovich-Popov lemma (KYP). The KYP lemma relates frequency-domain inequalities (FDIs) with a finite dimensional linear matrix inequality (LMI) [1]. The standard result of the KYP-lemma [2] can be stated as the equivalence between the FDI

$$\begin{bmatrix}(j\omega I-\mathsf{A})^{-1}\mathsf{B}\\ I\end{bmatrix}^* \Theta \begin{bmatrix}(j\omega I-\mathsf{A})^{-1}\mathsf{B}\\ I\end{bmatrix} \prec 0$$

for all $\omega \in \mathbb{R}$, and the LMI

$$\begin{bmatrix}\mathsf{A} & \mathsf{B}\\ I & 0\end{bmatrix}^* \begin{bmatrix}0 & P\\ P & 0\end{bmatrix} \begin{bmatrix}\mathsf{A} & \mathsf{B}\\ I & 0\end{bmatrix} + \Theta \prec 0.$$

Although the standard KYP lemma has proven to be a powerful tool in the context of state-space control theory, the lemma does not take into account finite-frequency specifications.

In the last decade, extensions of the standard result were proposed to deal with finite-frequency specifications. For instance, [3] developed a framework, the so-called generalized KYP lemma (gKYP), where the frequency interval in which the FDI holds is captured using the set

$$\Lambda(\Phi,\Psi) = \{\lambda \in \mathbb{C} : \sigma(\lambda,\Phi) = 0, \sigma(\lambda,\Psi) \geq 0\},$$

where $\sigma(\lambda,\Pi) = \begin{bmatrix}\lambda^* & 1\end{bmatrix} \Pi \begin{bmatrix}\lambda^* & 1\end{bmatrix}^*$. The definition of Φ and Ψ is given in [3].

Using the gKYP framework, one may deal with specifications in low, middle or high frequencies based on the equivalence between the FDI

$$\begin{bmatrix}(j\omega I-\mathsf{A})^{-1}\mathsf{B}\\ I\end{bmatrix}^* \Theta \begin{bmatrix}(j\omega I-\mathsf{A})\mathsf{B}\\ I\end{bmatrix} \prec 0,\ \forall \omega \in \Lambda(\Phi,\Psi)$$

and the LMI,

$$\begin{bmatrix}\mathsf{A} & \mathsf{B}\\ I & 0\end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix}\mathsf{A} & \mathsf{B}\\ I & 0\end{bmatrix} + \Theta \prec 0,$$

with $\mathsf{Q} = \mathsf{Q}^* \succ 0$ and $\mathsf{P} = \mathsf{P}^*$. A key element in the derivation of this equivalence is the S-procedure (the reader is referred to [3], [4] for further details).

Another extension in the literature was made in [5], where the authors used similar methodology and converted any low, middle or high frequency specifications into the feasibility of the pair of LMIs

$$\Theta + \mathrm{He}\left\{\begin{bmatrix}F\\ G\end{bmatrix}\begin{bmatrix}I - j\tilde{\omega}_i I\end{bmatrix}(T\otimes I)\begin{bmatrix}\mathsf{A} & \mathsf{B}\\ I & 0\end{bmatrix}\right\} \prec 0, \quad i = 1,2,$$

where $\tilde{\omega}_i$ is obtained from a bilinear transformation applied to $\Lambda(\Phi,\Psi)$ (see [5] for details).

The main contribution of this paper is to propose a necessary and sufficient condition to assess specifications in middle-frequency interval with an additional multiplier. This characteristic turns out to be more appropriate for control design problems, as shown in the sequel, where the state-feedback, observer-based estimation and filtering problems can be solved within the framework of the proposed condition. Although this approach is shown to furnish the optimal solution in terms of the $\mathscr{H}_\infty$ norm on a middle-frequency interval, the resulting controllers, observers and filters have complex-valued realizations and, therefore their mathematical representations cannot be physically implemented. An alternative procedure is presented in order to overcome this issue, at expense of losing the necessity of the conditions.

The paper is organized as follows. Section II presents the gKYP, specialized to compute the $\mathscr{H}_\infty$ norm on middle-frequency intervals, and the Projection lemma. Section III presents the proposed necessary and sufficient condition to test the feasibility of an FDI in middle-frequency intervals as well as solutions for the state-feedback, observer-based estimation and filtering problems. Section IV introduces the proposed methodology to cope with complex-valued realizations and stability requirements. Finally, Section V

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presents some design examples to illustrate the utility of the proposed condition.

The following notation is used. The imaginary number is given by $j=\sqrt{-1}$. The space of complex (real) rectangular matrices is represented by $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$). Given a matrix $F\in\mathbb{C}^{m\times n}$, its complex conjugate transpose and a basis for the null space of F (a full column rank matrix such that $FF^{\perp}=0$ and $\begin{bmatrix}F^* & F^{\perp}\end{bmatrix}$ has column rank equal to n), are denoted by F^* and $F^{\perp}$, respectively. The symbol $\mathscr{H}_n$ stands for the $n\times n$ set of hermitian matrices. $\mathrm{He}(F)$ is a short-hand notation for $F+F^*$ and $(F^{-1})^*=F^{-*}$. The symbol $\otimes$ means the Kronecker product.

II. Preliminaries

This section introduces some fundamental results. The following lemma presents a specialization of the generalized KYP-lemma (gKYP) [3] to compute an upper bound for the $\mathscr{H}_\infty$ norm on a middle-frequency interval[1].

Lemma *1 ($\mathscr{H}_\infty$ Middle Frequency):* Let matrices $\mathsf{A}\in\mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $\mathsf{B}\in\mathbb{C}^{n\times r}$, $\mathsf{C}\in\mathbb{C}^{p\times n}$ and $\mathsf{D}\in\mathbb{C}^{p\times r}$, and scalars ω_1, ω_2 be given. Then, the following statements are equivalent:

i) $\|H(s)\|_\infty<\gamma$, $\forall s=j\omega$ and $\omega\in[\omega_1,\omega_2]$, where

$$H(s)=\mathsf{C}(sI-\mathsf{A})^{-1}\mathsf{B}+\mathsf{D}. \tag{1}$$

ii) $\exists P,\ Q\in\mathscr{H}_n$ such that $Q\succ 0$ and the following inequality holds:

$$\begin{bmatrix}\mathsf{A} & \mathsf{B}\\ I & 0\end{bmatrix}^*\begin{bmatrix}-Q & P+j\omega_cQ\\ P-j\omega_cQ & -\omega_1\omega_2Q\end{bmatrix}\begin{bmatrix}\mathsf{A} & \mathsf{B}\\ I & 0\end{bmatrix} + \begin{bmatrix}\mathsf{C} & \mathsf{D}\\ 0 & I\end{bmatrix}^*\begin{bmatrix}I & 0\\ 0 & -\gamma^2 I\end{bmatrix}\begin{bmatrix}\mathsf{C} & \mathsf{D}\\ 0 & I\end{bmatrix}\prec 0 \tag{2}$$

where $\omega_c=(\omega_1+\omega_2)/2$.

Proof: See [4] or [3]. ■

As an important remark, [6] showed that, when matrices A, B, C and D are real, variables P and Q can be made real without loss of generality.

The Projection lemma (see [7]), briefly presented in the following, is an important tool in the development of this work.

Lemma *2 (Projection Lemma):* Given matrices $\mathscr{Q}\in\mathscr{H}_{n_x}$, $\mathscr{W}\in\mathbb{C}^{m_1\times n_x}$ and $\mathscr{Y}\in\mathbb{C}^{m_2\times n_x}$, the following statements are equivalent:

i) Inequalities $\mathscr{W}^{\perp^*}\mathscr{Q}\mathscr{W}^{\perp}\prec 0$ and $\mathscr{Y}^{\perp^*}\mathscr{Q}\mathscr{Y}^{\perp}\prec 0$ hold.

ii) There exists a matrix $\mathscr{X}\in\mathbb{C}^{m_1\times m_2}$ such that

$$\mathscr{Q}+\mathscr{W}^*\mathscr{X}\mathscr{Y}+\mathscr{Y}^*\mathscr{X}^*\mathscr{W}\prec 0 \tag{3}$$

Throughout the paper, the following assumption is enforced.

Assumption 1:

$$\mathsf{D}^*\mathsf{D}-\gamma^2 I\prec 0.$$

[1]Since the stability of A is not assumed a priori, some authors refer to γ as a bound to the L_∞ norm of the system.

Remark *1:* Notice that Assumption 1 is not a strong constraint. For instance, the above is automatically satisfied when $\mathsf{D}=0$.

III. Main Results

The following lemma, which provides an equivalent condition for (2) with an extra multiplier, can be considered as the base of the results presented in this paper.

Lemma *3:* Suppose that Assumption 1 holds. Let matrices $\mathsf{A}\in\mathbb{C}^{n\times n}$, $\mathsf{B}\in\mathbb{C}^{n\times r}$, $\mathsf{C}\in\mathbb{C}^{p\times n}$ and $\mathsf{D}\in\mathbb{C}^{p\times r}$, and scalars ω_1, ω_2 be given. Then, the following condition is equivalent to (2):

i) There exist matrices $\mathsf{P},\ \mathsf{Q}\in\mathscr{H}_n$ and $G\in\mathbb{C}^{n\times n}$ such that

$$\begin{bmatrix}-\mathsf{Q} & \mathsf{P}+j\omega_c\mathsf{Q}-G^* & 0 & 0\\ \star & -\omega_1\omega_2\mathsf{Q}+\mathrm{He}(G\mathsf{A}) & G\mathsf{B} & \mathsf{C}^*\\ \star & \star & -\gamma^2 I & \mathsf{D}^*\\ \star & \star & \star & -I\end{bmatrix}\prec 0. \tag{4}$$

Proof: The equivalence between (2) and (4) is established using Lemma 2. First, applying Schur's complement with respect to the $(4,4)$-block, note that (4) can be rewritten as:

$$\mathscr{Q}_1+\begin{bmatrix}0\\ I\\ 0\end{bmatrix}G\begin{bmatrix}-I & \mathsf{A} & \mathsf{B}\end{bmatrix}+\begin{bmatrix}-I\\ \mathsf{A}^*\\ \mathsf{B}^*\end{bmatrix}G^*\begin{bmatrix}0 & I & 0\end{bmatrix}\prec 0 \tag{5}$$

with

$$\mathscr{Q}_1=\begin{bmatrix}-\mathsf{Q} & \mathsf{P}+j\omega_c\mathsf{Q} & 0\\ \star & -\omega_1\omega_2\mathsf{Q}+\mathsf{C}^*\mathsf{C} & \mathsf{C}^*\mathsf{D}\\ \star & \star & \mathsf{D}^*\mathsf{D}-\gamma^2 I\end{bmatrix}.$$

Then, observe that

$$\mathscr{W}^{\perp^*}\mathscr{Q}_1\mathscr{W}^{\perp}\prec 0 \quad\Rightarrow\quad \begin{bmatrix}-\mathsf{Q} & 0\\ 0 & \mathsf{D}^*\mathsf{D}-\gamma^2 I\end{bmatrix}\prec 0 \tag{6}$$

and $\mathscr{Y}^{\perp^*}\mathscr{Q}_1\mathscr{Y}^{\perp}\prec 0$ is condition (2), with

$$\mathscr{W}^{\perp}=\begin{bmatrix}I & 0\\ 0 & 0\\ 0 & I\end{bmatrix},\quad \mathscr{Y}^{\perp}=\begin{bmatrix}\mathsf{A} & \mathsf{B}\\ I & 0\\ 0 & I\end{bmatrix}.$$

Condition (6) is automatically satisfied by Assumption 1 and, from the block $(1,1)$ of (4), $\mathsf{Q}\succ 0$. ■

Remark *2:* The above conditions could be extended to deal with the case of low-frequency interval using a change of variables (see [4] for details). In fact, in that case, complex-valued realizations would not be an issue, since the results would be necessary and sufficient with a real multiplier.

Remark *3:* Note that condition (4) does not have products involving the multiplier G and variables P or Q. Consequently, some control problems can be solved by taking advantage of this fact.

Remark *4:* An extension of the gKYP lemma to cope with the problem of full order dynamic controller design has been proposed in [8] using a slightly different approach.

A. State-Feedback

The state-feedback control problem can be stated as follows. Consider a linear time-invariant system whose state-space representation is given by:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t)\\ z(t) &= C_z x(t) + D_{zu} u(t) + D_{zw} w(t),\end{aligned} \tag{7}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control signal, $w \in \mathbb{R}^r$ is the disturbance signal and $z \in \mathbb{R}^p$ is the regulated output. The goal is to design a control law $u = Kx$ such that the $\mathscr{H}_\infty$ norm from the disturbance signal w to the output z is minimized on the frequency interval $\omega \in [\omega_1, \omega_2]$. In other words, find a gain K such that

$$\|H(j\omega)\|_\infty < \gamma, \ \forall \omega \in [\omega_1, \omega_2], \tag{8}$$

where,

$$H(s) = (C_z + D_{zu}K)(sI - A - B_u K)^{-1} B_w + D_{zw}. \tag{9}$$

Using the result of Lemma 3, this problem can be formulated in terms of convex optimization, as shown in the next theorem.

Theorem *1:* Consider a linear time-invariant system given by (7) and suppose that Assumption 1 holds. Let scalars ω_1, ω_2 be given. Then, the following statements are equivalent:

i) There exists a gain K that fulfills (8) for $H(s)$ given in (9).
ii) There exist matrices P, $Q \in \mathscr{H}_n$ and matrices $H \in \mathbb{C}^{r\times n}$ and $Z \in \mathbb{C}^{n\times n}$ such that

$$\begin{bmatrix} -Q & P + j\omega_c Q - Z^* & 0 & 0\\ \star & -\omega_1\omega_2 Q + \mathrm{He}(AZ + B_u H) & B_w & \mathscr{P}_{24}\\ \star & \star & -\gamma^2 I & D_{zw}^*\\ \star & \star & \star & -I \end{bmatrix} \prec 0, \tag{10}$$

with

$$\mathscr{P}_{24} = Z^* C_z^* + H^* D_{zu}^*.$$

In the affirmative case, the gain K is given by $K = HZ^{-1}$.

Proof: Firstly, note that inequality (10) can be written as:

$$T_1^* \left\{ \mathscr{Q}_2 + \mathrm{He}\left(\begin{bmatrix} 0\\ Z^{-*}\\ 0\\ 0 \end{bmatrix} \begin{bmatrix} -I & A + B_u K & B_w & 0 \end{bmatrix} \right) \right\} T_1 \prec 0 \tag{11}$$

where $T_1 = \mathrm{diag}\{Z^*, Z^*, I, I\}$,

$$\mathscr{Q}_2 = \begin{bmatrix} Z^{-*}QZ^{-1} & Z^{-*}(P + j\omega_c Q - Z^*)Z^{-1} & 0 & 0\\ \star & Z^{-*}(-\omega_1\omega_2 Q)Z^{-1} & 0 & \mathscr{X}_{24}\\ \star & \star & -\gamma^2 I & D_{zw}^*\\ \star & \star & \star & -I \end{bmatrix},$$

and $\mathscr{X}_{24} = C_z^* + K^* D_{zu}^*$.

Therefore, defining $G = Z^{-*}$, $\mathsf{Q} = Z^{-*}QZ^{-1}$, $\mathsf{P} = Z^{-*}PZ^{-1}$, $\mathsf{A} = A + B_u K$, $\mathsf{B} = B_w$, $\mathsf{C} = C_z + D_{zu}K$ and $\mathsf{D} = D_{zw}$, and applying a congruence transformation, inequality (11) is shown to be equivalent to (4). Then, the result follows from Lemma 3. ∎

B. Observer-Based Estimation

In this section, the dual problem of the state feedback control is considered, i.e., the so-called observer design, that can be stated as follows. Consider a linear time-invariant system with the following state-space representation:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_w w(t)\\ z(t) &= C_z x(t) + D_{zw} w(t)\\ y(t) &= C_y x(t) + D_{yw} w(t),\end{aligned} \tag{12}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^r$ is the disturbance input, $z \in \mathbb{R}^p$ is the output to be estimated and $y \in \mathbb{R}^q$ is the measured output. Let the state-space representation for the observer be given by

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + F(y(t) - \hat{y}(t))\\ \hat{y}(t) &= C_y \hat{x}(t)\\ \hat{z}(t) &= C_z \hat{x}(t).\end{aligned}$$

The purpose is to find a matrix F in order to minimize the $\mathscr{H}_\infty$ norm of the transfer matrix from w to $\tilde{z} = C_z e$, where $e = x - \hat{x}$, in the middle-frequency interval given by $\omega \in [\omega_1, \omega_2]$. The dynamics of the error is given by

$$\dot{e}(t) = (A - FC_y)e(t) + (B_w - FD_{yw})w(t), \tag{13}$$

and the transfer matrix from w to $\tilde{z}$ by

$$H(s) = C_z(sI - A + FC_y)^{-1}(B_w - FD_{yw}) + D_{zw}. \tag{14}$$

Therefore, the observer-based estimation problem can be alternatively stated as: find a gain F that minimizes the $\mathscr{H}_\infty$ norm of $H(s)$ given in (14) in the frequency interval $\omega \in [\omega_1, \omega_2]$.

Similarly to the state-feedback case, the search for the gain F can be converted into a convex problem, as presented in the next theorem.

Theorem *2:* Consider a linear time-invariant system given by (12) and suppose that Assumption 1 holds. Let scalars ω_1, ω_2 be given. Then, the following statements are equivalent:

i) There exists a gain F such that (8) holds for $H(s)$ given in (14).
ii) There exist matrices P, $Q \in \mathscr{H}_n$ and matrices $H \in \mathbb{C}^{n\times q}$ and $Z \in \mathbb{C}^{n\times n}$ such that

$$\begin{bmatrix} -Q & P + j\omega_c Q - Z^* & 0 & 0\\ \star & -\omega_1\omega_2 Q + \mathrm{He}(ZA - HC_y) & ZB_w - HD_{yw} & C_z^*\\ \star & \star & -\gamma^2 I & D_{zw}^*\\ \star & \star & \star & -I \end{bmatrix} \prec 0. \tag{15}$$

In the affirmative case, the gain F is given by $F = HZ^{-1}$.

Proof: Using the conjugate transpose of the transfer matrix (14), the result follows from the state-feedback condition by a duality argument. ∎

C. Filtering Problem

Consider an asymptotically stable linear time-invariant system with state-space representation given by:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_w w(t)\\ z(t) &= C_z x(t) + D_{zw} w(t)\\ y(t) &= C_y x(t) + D_{yw} w(t),\end{aligned} \tag{16}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^r$ is the noise input, $z \in \mathbb{R}^p$ is the output to be estimated and $y \in \mathbb{R}^q$ represents the measured output. The aim is to design a full order proper filter with state-space representation

$$\begin{aligned} \dot{x}_f(t) &= A_f x_f(t) + B_f y(t) \\ \hat{z}(t) &= C_f x_f(t) + D_f y(t), \end{aligned} \tag{17}$$

in order to minimize the $\mathscr{H}_\infty$ norm of the transfer matrix from w to $e = z - \hat{z}$,

$$H(s) = \mathsf{C}(sI - \mathsf{A})^{-1}\mathsf{B} + \mathsf{D}, \tag{18}$$

where,

$$\left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B_w \\ B_f C_y & A_f & B_f D_{yw} \\ \hline C_z - D_f C_y & -C_f & D_{zw} - D_f D_{yw} \end{array}\right], \tag{19}$$

for $s = j\omega$, $\omega \in [\omega_1, \omega_2]$.

In a similar way, the filtering problem can be formulated as a convex problem, as shown in the next theorem.

Theorem *3:* Consider a linear time-invariant system given as in (16) and suppose that Assumption 1 holds. Then, the following conditions are equivalent:

i) There exists a filter with state-space representation given by (17) such that $\|H(j\omega)\|_\infty < \gamma, \forall \omega \in [\omega_1, \omega_2]$, where $H(s)$ is given by (18).

ii) There exist matrices $G_{11}, G_{21}, \hat{K}, M_{Af}, P_{12}, Q_{12} \in \mathbb{C}^{n\times n}$, $M_{Bf} \in \mathbb{C}^{n\times r}$, $C_f \in \mathbb{R}^{p\times n}$ and $D_f \in \mathbb{R}^{p\times r}$, and hermitian matrices $P_{11}, P_{22}, Q_{11}, Q_{22} \in \mathscr{H}_n$ such that inequality (21) holds. In addition, matrices A_f and B_f of the filter are given by

$$A_f = \hat{K}^{-1} M_{Af}, \quad B_f = \hat{K}^{-1} M_{Bf}. \tag{20}$$

Proof: Note that condition (21) can be written as:

$$\begin{bmatrix} -\mathsf{Q} & \mathsf{P} + j\omega_c \mathsf{Q} & 0 & 0 \\ \star & -\omega_1\omega_2 \mathsf{Q} & 0 & \mathsf{C}^* \\ \star & \star & -\gamma^2 I & \mathsf{D}^* \\ \star & \star & \star & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} 0 \\ \mathsf{G} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -I & \mathsf{A} & \mathsf{B} & 0 \end{bmatrix} \right\} \prec 0 \tag{23}$$

where

$$\mathsf{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}, \quad \mathsf{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}, \quad \mathsf{G} = \begin{bmatrix} G_{11} & \hat{K} \\ G_{21} & \hat{K} \end{bmatrix},$$

and A, B, C and D are given in (19). Performing a change of coordinates on the filter realization, as proposed in [9], it is possible to show that the choice of the blocks $G_{12} = G_{22} = \hat{K}$ can be made without loss of generality. Consequently, the equivalence between i) and ii) is established using Lemma 3. ■

IV. Enforcing Stability and Realness

The results presented in Theorems 1, 2 and 3 are necessary and sufficient conditions to establish middle-frequency specifications for the $\mathscr{H}_\infty$ norm of the corresponding closed-loop systems (provided Assumption 1 holds). However, these conditions have two major drawbacks. Firstly, the gain K (F) is complex, resulting in a state-feedback controller (observer) that cannot be physically implemented. The same occurs with matrices A_f and B_f in the filtering problem. Secondly, the conditions have not assured that the closed-loop matrices, A, are asymptotically stable.

The realness of the controller and filter matrices is addressed by constraining all optimization variables associated with the multiplier G to be real. For instance, Z and H in the state-feedback control and observer problems, and $\hat{K}$, M_{Af}, M_{Bf} in the filtering problem. Note that all other variables, such as P, Q, G_{11} and G_{21}, can remain complex.

The stability of the closed-loop system in the state feedback, observer and filtering problems can be circumvented by requiring asymptotically stability for the closed-loop dynamic matrix. This is done by adding inequalities

$$\begin{bmatrix} \xi \mathrm{He}(Z) & W + \xi(Z^*A^* + H^*B_u^*) - Z \\ \star & \mathrm{He}(AZ + B_u H) \end{bmatrix} \prec 0, \quad W \succ 0 \tag{24}$$

for the state-feedback. The rationale behind this strategy is similar to the one described in [8, Lemma 5]. Indeed, if (24) is satisfied, then multiplication on the left by $\begin{bmatrix} A + B_u K & I \end{bmatrix}$ and on the right by its conjugate transpose yields

$$(A + B_u K)W + W(A + B_u K)^* \prec 0,$$

which implies closed-loop stability. Similarly, the extra constraints

$$\begin{bmatrix} -\xi \mathrm{He}(Z) & W + \xi(ZA - HC_y) - Z \\ \star & \mathrm{He}(ZA - HC_y) \end{bmatrix} \prec 0, \quad W \succ 0 \tag{25}$$

for the observer problem, and (22) and

$$W = \begin{bmatrix} W_{11} & W_{12} \\ \star & W_{22} \end{bmatrix} \succ 0,$$

for the filtering problem enforce stability. Note that the additional conditions are LMIs for fixed values of $\xi > 0$.

Although the main drawbacks regarding implementation purposes have been surpassed, the proposed conditions with the additional constraints become sufficient only.

Remark *5:* Note that variables L_{11}, L_{21}, G_{11} and G_{21} in inequalities (21) and (22) can be eliminated without loss of generality using a projection argument, similar to the one used in [10].

V. Filter Design Examples

In this section, some examples are presented in order to evaluate the proposed conditions. In addition, the inherent conservatism due to the constraints mentioned in the previous section is investigated. The routines were implemented in MATLAB, version 8.2.0.701 64 bits, using YALMIP [11] and Mosek [12].

$$
\begin{bmatrix}
-Q_{11} & -Q_{12} & P_{11}+jQ_{11}\omega_c-G_{11}^{*} & P_{12}+jQ_{12}\omega_c-G_{21}^{*} & 0 & 0 \\
\star & -Q_{22} & P_{12}^{*}+jQ_{12}^{*}\omega_c-\hat{K}^{*} & P_{22}+jQ_{22}\omega_c-\hat{K}^{*} & 0 & 0 \\
\star & \star & -\omega_1\omega_2 Q_{11}+\mathrm{He}(G_{11}A+M_{Bf}C_y) & -\omega_1\omega_2 Q_{12}+M_{Af}+A^{*}G_{21}^{*}+C_y^{*}M_{Bf}^{*} & G_{11}B_w+M_{Bf}D_{yw} & C_z^{*}-C_y^{*}D_f^{*} \\
\star & \star & \star & -\omega_1\omega_2 Q_{22}+\mathrm{He}(M_{Af}) & G_{21}B_w+M_{Bf}D_{yw} & -C_f^{*} \\
\star & \star & \star & \star & -\gamma^2 I & D_{zw}^{*}-D_{yw}^{*}D_f^{*} \\
\star & \star & \star & \star & \star & -I
\end{bmatrix} \prec 0 \tag{21}
$$

$$
\begin{bmatrix}
-\xi\mathrm{He}(L_{11}) & -\xi(\hat{K}+L_{21}^{*}) & W_{11}+\xi(L_{11}A+M_{Bf}C_y)-L_{11} & W_{12}+\xi M_{Af}-\hat{K} \\
\star & -\xi\mathrm{He}(\hat{K}) & W_{12}^{*}+\xi(L_{21}A+M_{Bf}C_y)-L_{21} & W_{22}+\xi M_{Af}-\hat{K} \\
\star & \star & \mathrm{He}(L_{11}A+M_{Bf}C_y) & M_{Af}+A^{*}L_{21}^{*}+C_y^{*}M_{Bf}^{*} \\
\star & \star & \star & \mathrm{He}(M_{Af})
\end{bmatrix} \prec 0 \tag{22}
$$

In the following numerical experiments, ξ is chosen in the following set (as in other papers that deal with scalars in LMIs for robust control [13]):

$$
\Xi = \left\{\xi \in \mathbb{R} : \xi = 10^{i}, \quad \text{for} \quad i = -6, -5, \ldots, 5, 6\right\}.
$$

Example *1:* Consider an unstable linear time-invariant system as in (7) with matrices (randomly generated) given by

$$
\left[\, A \,\vdots\, B_u \,\vdots\, B_w \,\right] = \left[\begin{array}{ccc:c:c} 2.27 & 0.44 & 1.08 & -0.2 & -2.80 \\ -0.05 & -0.91 & 0.31 & -0.15 & 0.39 \\ -1.55 & 0.05 & 0.30 & -0.1 & 0.99 \end{array}\right],
$$

$$
\left[\, C_z \,\vdots\, D_{zu} \,\vdots\, D_{zw} \,\right] = \left[\begin{array}{ccc:c:c} -1.3 & 0.62 & -1.68 & -0.65 & -1.29 \\ -1.52 & -1.51 & 0.79 & 1.24 & -0.61 \end{array}\right].
$$

The aim is the design of a state-feedback controller in the view of Theorem 1. In order to evaluate the conservativeness of the result, three different frequency-range specifications are considered and the conservatism of the additional constraints (see Section IV) is analyzed by comparing the value of the norm associated with the unconstrained solution, i.e., the value of the norm provided by the complex-valued controller without the stability requirement. The results are summarized in Table I. The second column presents the value of the norm with a complex controller and without the stability requirement (CWS). On the third and fourth columns the stability requirement for complex (CAS) and real controller (RV) is added, respectively. As can be seen, the value of the norm for the complex controller without stability requirement is less than in the other cases, which is expected since a broader class of systems is allowed. The interesting trend of this example is to show the difference between the third and fourth columns. The pronounced difference shows the conservatism introduced by forcing variables Z and H to be real.

Example *2:* In this example, the designs of an observer and a filter for an asymptotically stable system are analyzed. Consider the following linear time-invariant system as in (12) (or, as in (16)) with $D_{zw} = 0_{p\times r}$, whose matrices are given by:

$$
\left[\begin{array}{c:c} A & B_w \\ \hdashline C_z & 0_{p\times r} \\ \hdashline C_y & D_{yw} \end{array}\right] = \left[\begin{array}{cc:cc} -0.6 & 4 & 0 & 0 \\ -4 & -0.6 & 1.5 & 0 \\ \hdashline 0 & 1 & 0 & 0 \\ \hdashline 0 & -1.2 & 0 & 1 \end{array}\right]. \tag{26}
$$

TABLE I

$\mathscr{H}_\infty$ NORMS FOR EXAMPLE 1 USING THEOREM 1 FOR DIFFERENT MIDDLE-FREQUENCY SPECIFICATIONS.

Interval	CWS	CAS	RV
$\omega \in [0.01, 1]$	3.85	9.88	10.34
$\omega \in [1, 10]$	3.83	14.43	21.53
$\omega \in [50, 100]$	1.43	1.46	21.53

The results obtained by applying Theorems 2 and 3 for system (26) with the frequency specification $[\omega_1, \omega_2] = [10, 100]$ are shown in Table II. Note that, as expected, the results for the observer comply with those from Example 1, i.e., the conservativeness is increased when the stability requirement is added, and further raised with the imposition of real parameters. On the other hand, the constrained filtering formulation yields the optimal filter norm, irrespective of the additional constraints. This behavior would suggest that no loss of generality was imposed in the filter design, an issue still under investigation.

TABLE II

OBSERVER AND FILTER $\mathscr{H}_\infty$ PERFORMANCE COMPARISON FOR THE SYSTEM OF EXAMPLE 2.

$\omega \in [10, 100]$	CWS	CAS	RV
$\mathscr{H}_\infty$*(Observer)*	0.17	0.29	0.70
$\mathscr{H}_\infty$*(Filter)*	0.17	0.17	0.17

Example *3:* The proposed conditions for filter design can also cope with uncertain systems. To illustrate this, consider the uncertain linear time-invariant system (also investigated in [14], [15]) given by

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -0.6 & 4+\delta \\ -4 & -0.6 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1.5 & 0 \end{bmatrix} w(t) \\
z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \\
y(t) &= \begin{bmatrix} 0 & -1.2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(t),
\end{aligned} \tag{27}
$$

where $|\delta| \leq 1$. Note that this representation can be converted into a polytopic model with two vertices. The condition of Theorem 3 can be extended to deal with this model using LMI relaxations. For the sake of simplicity, matrices P and Q are allowed to be affine-dependent on the uncertain parameter while all other decision variables are considered as constant matrices. This is done by adding a subscript i in each matrix of the system and in the block matrices that form P and Q, and test LMIs (21) and (22) for $i = 1, \ldots, N$, where N is the number of vertices (in this case $N = 2$), similarly to the robust control design technique with affine parameter dependent Lyapunov matrix proposed in [16] (see also [17]). The value of γ provided through this procedure is an upper bound for the worst-case $\mathscr{H}_\infty$ norm on the frequency-interval $\omega \in [\omega_1, \omega_2]$.

Figure 1 shows the upper bound provided by the condition of Theorem 3 (green line) for the frequency specification $\omega \in [10, 100]$, which is $\gamma = 0.1821 = -14.8$dB. The singular value diagram is also shown for different values of the uncertain parameter (a fine grid was applied) of the polytopic model obtained from (27) connected with the designed filter.

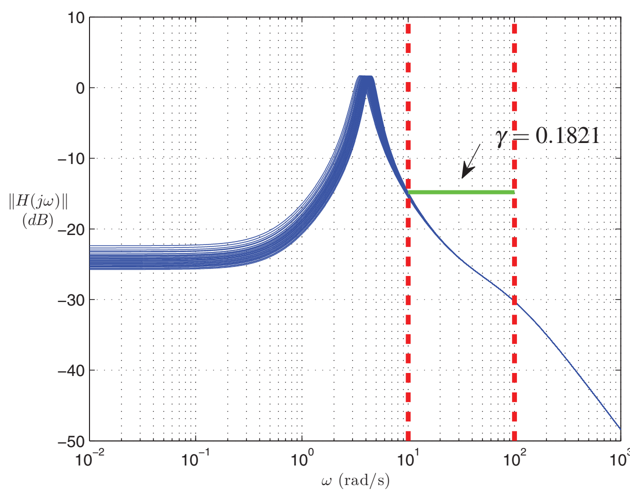

Fig. 1. Singular value diagram for different values of the uncertain parameter for the uncertain model (27) (blue lines) and the upper bound provided by Theorem 3 (green line) for the worst case $\mathscr{H}_\infty$ norm in the frequency interval $\omega \in [10, 100]$ (between dashed red lines).

VI. Conclusion

This paper has dealt with three different control problems using extensions of the generalized Kalman-Yakubovich-Popov lemma for middle-frequency specifications. Solutions for the state-feedback, observer-based estimation and filtering problems were provided using complex multipliers. The drawbacks associated to the complex multiplier are evident, since the synthesized controllers, observers and filters are not implementable. To surpass this issue, the multiplier was imposed to be real and the conservatism of this constraint was analyzed by means of numerical examples.

The examples suggest that the constraints can be imposed in the filtering problem formulation to cope with middle-frequency specifications without loss of generality, since the optimal solution was obtained in all the numerical examples investigated. Moreover, robust filters for uncertain systems in polytopic domains can be obtained from a straightforward extension of the proposed condition.

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