2018 Annual American Control Conference (ACC)
June 27–29, 2018. Wisconsin Center, Milwaukee, USA

$\mathscr{H}_\infty$ filter design with low- and middle-frequency specifications for continuous-time linear systems: LMI conditions derived from two different extensions of the KYP lemma

Lício B. R. R. Romão[1], Maurício C. de Oliveira[2], Pedro L. D. Peres[1], Ricardo C. L. F. Oliveira[1]

***Abstract*— This paper addresses the full-order filtering problem with low- and middle-frequency $\mathscr{H}_\infty$ specifications for continuous-time linear systems. Two distinct extensions of the KYP Lemma are used to produce synthesis conditions capable to cope with the frequency specifications. A comparison between the results obtained with the two extensions in terms of numerical complexity and the bound provided for the $\mathscr{H}_\infty$ norm is made by means of numerical examples. The design conditions, given in the form of linear matrix inequalities, can also cope with polytopic time-invariant linear systems. Two main contributions distinguish this paper from previous results: *i)* a simpler condition to assure asymptotic stability of the filtered system; *ii)* a new synthesis condition based on an alternative extension of the KYP Lemma.**

I. INTRODUCTION

The Kalman-Yakubovich-Popov Lemma (KYP) is one of the most important theoretical results in control theory. By establishing an equivalence between frequency domain inequalities and a linear matrix inequality (LMI), the KYP Lemma can be used to check, for instance, positive realness or boundedness of a rational transfer function by means of a convex optimization problem. A simple proof of this lemma can be found in [1].

Besides providing theoretical foundation to solve several control problems, the standard KYP Lemma cannot cope with finite-frequency specifications. In this direction, some extensions of the KYP Lemma have been proposed in the literature to deal with these requirements. First, [2] established an LMI-based necessary and sufficient condition to verify whether a frequency inequality holds in segments of imaginary axis and of the unit circle, result known as the generalized KYP (gKYP) Lemma. Some improvement and extra generalizations for the gKYP Lemma were presented in [3] and [4]. Second, [5] provided another extension of the KYP Lemma using a slightly different approach. Actually, instead of using the S-procedure as in [2], this last extension uses projection arguments to prove the equivalence between a new set of LMI conditions (including slack variables) and the original frequency-domain inequalities.

These aforementioned extensions paved the way for the development of new LMI conditions addressing classical problems in control theory with $\mathscr{H}_\infty$ frequency specifications, such as state-feedback control, the observer-based estimation and filter design [6], [7], [8]. Particularly in the context of full-order filtering, the main drawback is that the available conditions are only sufficient, even for precisely known systems (not affected by uncertainties). In some situations it is possible to prove the necessity, as in [8] for continuous-time systems with middle-frequency specification, but the filter realization is given in terms of complex matrices and might not be asymptotically stable.

This paper furthers into the design of full-order $\mathscr{H}_\infty$ filters with frequency specification. The methodology uses the extensions for the KYP Lemma proposed in [2] and in [5] to develop different synthesis conditions for the filtering problem, comparing the advantages and disadvantages in terms of numerical complexity and accuracy of the $\mathscr{H}_\infty$ guaranteed costs, specially when the systems to be filtered are affected by polytopic uncertainties. The main contributions of this paper are twofold: *i)* a new approach to ensure asymptotic stability for the filtered system; *ii)* a design condition for the filtering problem using the extension for the KYP proposed in [5]. Numerical examples are presented to illustrate the results, including a comparison with the condition in [7].

This paper is organized as follows. Section II states the filtering problem with finite-frequency specification. Section III shows the basic results and preliminaries that are necessary for the presentation of the proposed results. Section IV establishes two synthesis procedures in terms of LMIs and also presents a discussion about imposing asymptotic stability for the designed filter. Section V presents numerical comparisons of the proposed synthesis conditions. Finally, Section VI presents the final comments and conclusions.

The notation used throughout this paper is standard. Real and complex vectors spaces of dimension $n \times m$ are denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$, respectively. For a matrix $A \in \mathbb{C}^{n\times n}$, A^* represents its conjugate transpose (transpose for real matrices). The operator $\mathrm{He}(A) = A + A^*$ is used to shorten formulas. For hermitian (symmetric) matrices, $A \succ 0$ ($A \prec 0$) means that A is positive (negative) definite. A symmetric term in a matrix defined by blocks is denoted by $\star$. The identity (zero) matrix is denoted by I (0). The Kronecker product between matrices A and B is denoted by $A \otimes B$. The notation $\mathbf{min}_X\, f(X)$ **s. t.** $g(X) \prec 0$ indicates an optimization problem, i.e., minimize $f(X)$ subject to $g(X)$ on the cone of negative definite matrices.

II. PROBLEM FORMULATION

For simplicity, the conditions are developed for precisely known linear systems in terms of LMIs. The extension

Supported by the Brazilian agencies CAPES, CNPq and FAPESP (Proc. 2014/06408-4 and 2015/13135-7).

[1]School of Electrical and Computer Engineering, University of Campinas – UNICAMP, 13083-852, Campinas, SP, Brazil. {licio, ricfow, peres}@dt.fee.unicamp.br.

[2]Dept. of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093, USA. mauricio@ucsd.edu.

978-1-5386-5428-6/$31.00 ©2018 AACC

to cope with uncertain state space matrices in polytopic domains is immediate (see Section V).

Consider the asymptotically stable linear time-invariant system with state-space representation given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_w w(t) \\ z(t) &= C_z x(t) + D_{zw} w(t) \\ y(t) &= C_y x(t) + D_{yw} w(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^r$ is a exogenous input, $z \in \mathbb{R}^p$ is the output to be estimated, and $y \in \mathbb{R}^q$ is the measured output.

The goal is to design an asymptotically stable, full-order filter, whose state-space representation is given by

$$\begin{aligned} \dot{x}_f(t) &= A_f x_f(t) + B_f y(t) \\ z_f(t) &= C_f x_f(t) + D_f y(t), \end{aligned} \tag{2}$$

where $x_f \in \mathbb{R}^n$ and $z_f \in \mathbb{R}^p$, such that the $\mathscr{H}_\infty$ norm of the transfer matrix from the exogenous input w to the error $e = z - z_f$ is minimized in either low- or middle-frequency intervals. In other words, given the transfer matrix from w to e

$$H(s) = \mathsf{C}(sI - \mathsf{A})^{-1}\mathsf{B} + \mathsf{D}, \tag{3}$$

with

$$\left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B_w \\ B_f C_y & A_f & B_f D_{yw} \\ \hline C_z - D_f C_y & -C_f & D_{zw} - D_f D_{yw} \end{array}\right], \tag{4}$$

the problem can be reformulated as: find matrices (A_f, B_f, C_f, D_f), with A_f Hurwitz, such that $\|H(s)\|_\infty < \gamma, \forall s = j\omega$, where $\omega \in [0, \omega_\ell]$ for low-frequency specification or $\omega \in [\omega_1, \omega_2]$, $0 < \omega_1 < \omega_2$, for middle-frequency specification.

To ease the presentation, the filtering problem with low- and middle-frequency specifications are referenced from this point by LF and MF filtering, respectively.

III. Basic Results

Extensions of the standard KYP Lemma are the basis for obtaining a solution for the LF and MF filtering problems. In this section some preliminary results related to these extensions are presented. The reader is refereed to [9], [2], [5] for more details.

First and foremost, it is important to notice that there exist a homeomorphism almost everywhere between the curve[1]

$$\Lambda(\Phi, \Psi) = \left\{ s \in \mathbb{C} : \begin{bmatrix} s \\ 1 \end{bmatrix}^* \Phi \begin{bmatrix} s \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} s \\ 1 \end{bmatrix}^* \Psi \begin{bmatrix} s \\ 1 \end{bmatrix} \geq 0 \right\} \tag{5}$$

on the s-plane and the curve

$$\Lambda(\Phi_0, \Psi_0) = \left\{ v \in \mathbb{C} : \begin{bmatrix} v \\ 1 \end{bmatrix}^* \Phi_0 \begin{bmatrix} v \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} v \\ 1 \end{bmatrix}^* \Psi_0 \begin{bmatrix} v \\ 1 \end{bmatrix} \geq 0 \right\}, \tag{6}$$

where

$$\Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Psi_0 = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}, \quad \alpha \leq \gamma, \tag{7}$$

[1] In [2], necessary and sufficient conditions for this set not be a trivial or empty curve are presented.

on the v-plane. That is, there exits a continuous invertible map $\mathscr{T} : \mathbb{C} \to \mathbb{C}$ defined as

$$\mathscr{T}(s) = \frac{b - ds}{cs - a},$$

where the coefficients a, b, c and d are the entries of the matrix $T \in \mathbb{C}^{2\times 2}$ such[2] that

$$\Phi = T^* \Phi_0 T, \quad \Psi = T^* \Psi_0 T, \quad T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The continuous inverse of $\mathscr{T}(s)$, called $\mathscr{T}^{-1}(v)$, is defined as

$$\mathscr{T}^{-1}(v) = \frac{av + b}{cv + d},$$

almost everywhere. References to the coefficients of the map $\mathscr{T}(s)$ and to the parameters of equation (7) are made from this point on without warning.

The following lemma states a test to verify whether a frequency domain inequality holds by means of semidefinite programming, being central for the results presented in this paper.

Lemma *1 (see [5]):* Let $\Phi \in \mathbb{C}^{2\times 2}, \Psi \in \mathbb{C}^{2\times 2}, \Theta \in \mathbb{C}^{n+r}$, $\mathsf{A} \in \mathbb{R}^{n\times n}$ and $\mathsf{B} \in \mathbb{R}^{n\times r}$, with $\det(sI - \mathsf{A}) \neq 0$, $\forall s \in \Lambda(\Phi, \Psi)$ be given. The following conditions are equivalent:

i) The frequency domain inequality

$$\begin{bmatrix} (sI - \mathsf{A})^{-1}\mathsf{B} \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (sI - \mathsf{A})^{-1}\mathsf{B} \\ I \end{bmatrix} \prec 0$$

holds $\forall s \in \Lambda(\Phi, \Psi)$.

ii) There exist matrices $0 \prec Q = Q^* \in \mathbb{C}^{n\times n}$, $P = P^* \in \mathbb{C}^{n\times n}$, such that

$$\begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \end{bmatrix}^* (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \end{bmatrix} + \Theta \prec 0.$$

iii) There exists matrices $F \in \mathbb{C}^{n\times n}$ and $G \in \mathbb{C}^{r\times n}$ such that

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & j\tilde{\omega}_i I \end{bmatrix} (T \otimes I) \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \end{bmatrix} \right\} + \Theta \prec 0, \ i = 1, 2,$$

with $\tilde{\omega}_1 = -|\gamma/\alpha|^{1/2}$ and $\tilde{\omega}_2 = |\gamma/\alpha|^{1/2}$.

Since this paper is concerned with the minimization of the $\mathscr{H}_\infty$ norm in frequency intervals, matrix Θ is fixed as

$$\begin{bmatrix} \mathsf{C}^T\mathsf{C} & \mathsf{C}^T\mathsf{D} \\ \mathsf{D}^T\mathsf{C} & \mathsf{D}^T\mathsf{D} - \gamma^2 I \end{bmatrix}$$

and condition *i*) becomes $\|H(s)\|_\infty < \gamma, \forall s \in \Lambda(\Phi, \Psi)$.

At this point, it is worthwhile noting that although the condition *iii*) of Lemma 1 is already suitable for full-order filtering purposes, condition *ii*) still requires some manipulation. First, using the above choice for Θ, condition *ii*) can be rewritten as

$$\begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \\ \mathsf{C} & \mathsf{D} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} (\Phi \otimes P + \Psi \otimes Q) & 0 \\ 0 & \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \\ \mathsf{C} & \mathsf{D} \\ 0 & I \end{bmatrix} \prec 0.$$

[2] The existence of such a T has been proved in [2], provided that the set (6) represents a curve.

Then, after some algebraic manipulations, one can note that condition *ii*) is equivalent to

$$\begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \mathbf{R}_{11} & \begin{bmatrix} 0 \\ \mathsf{C}^T\mathsf{D} \end{bmatrix} \\ \begin{bmatrix} 0 & \mathsf{D}^T\mathsf{C} \end{bmatrix} & \mathsf{D}^T\mathsf{D} - \gamma^2 I \end{bmatrix} \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ I & 0 \\ 0 & I \end{bmatrix} \prec 0, \quad (8)$$

with

$$\mathbf{R}_{11} = (\Phi \otimes P + \Psi \otimes Q) + \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{C}^T\mathsf{C} \end{bmatrix}.$$

Hence, an equivalent condition to *ii*) can be obtained by means of the Finsler's Lemma [10], [11]. This result is presented in the next lemma.

Lemma *2:* Given matrices $\Phi \in \mathbb{C}^{2\times 2}, \Psi \in \mathbb{C}^{2\times 2}$, $\Theta \in \mathbb{C}^{n+r}$, $\mathsf{A} \in \mathbb{R}^{n\times n}$ and $\mathsf{B} \in \mathbb{R}^{n\times r}$, with $\det(sI - \mathsf{A}) \neq 0$, $\forall s \in \Lambda(\Phi, \Psi)$. The following condition is equivalent to the conditions of Lemma 1:

i) There exist matrices $0 \prec Q = Q^* \in \mathbb{C}^{n\times}, P = P^* \in \mathbb{C}^{n\times n}, L \in \mathbb{C}^{n\times n}, M \in \mathbb{C}^{n\times n}$ and $N \in \mathbb{C}^{r\times n}$ such that

$$\begin{bmatrix} (\Phi \otimes P + \Psi \otimes Q) & 0 & \begin{bmatrix} 0 \\ \mathsf{C}^T \end{bmatrix} \\ 0 & -\gamma^2 I & \mathsf{D}^T \\ \begin{bmatrix} 0 & \mathsf{C} \end{bmatrix} & \mathsf{D} & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} L \\ M \\ N \\ 0 \end{bmatrix} \begin{bmatrix} -I & \mathsf{A} & \mathsf{B} & 0 \end{bmatrix} \right\} \prec 0.$$

Proof: The proof follows from the discussion that precedes the lemma, and it is omitted for brevity. ∎

Remark *1:* Note that when the 2×2 block

$$\begin{bmatrix} -\gamma^2 I & \mathsf{D}^T \\ \mathsf{D} & -I \end{bmatrix}$$

is negative definite, variable N can be zeroed without loss of generality (this result is a consequence of the projection lemma [10]). Indeed, by a Schur's complement argument this block is negative definite if and only if inequality $\mathsf{D}^T\mathsf{D} - \gamma^2 I \prec 0$ holds. This is not a strong assumption (e.g., it would be trivially satisfied when $\mathsf{D} = 0$), so it is assumed to be true throughout this paper.

As showed in the beginning of this section, the choices for matrices Φ and Ψ define a curve on the complex plane. Considering that this paper copes with LF and MF specifications, the values for Φ and Ψ that are important for the results are

$$\Phi = \Phi_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$\Psi = \Psi_1 = \begin{bmatrix} -1 & 0 \\ 0 & \omega_\ell^2 \end{bmatrix}, \quad \Psi = \Psi_2 = \begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_1\omega_2 \end{bmatrix}.$$

The pair (Φ_c, Ψ_1) generates a low-frequency interval $[0, \omega_\ell]$; and the pair (Φ_c, Ψ_2) produces a middle-frequency interval of the form $[\omega_1, \omega_2]$ (See [2] for more details).

Having introduced the necessary framework, the next section presents the extensions of the conditions of Lemmas 1 and 2 for solving the LF and MF frequency filtering problems.

IV. Synthesis conditions

The conditions of Lemmas 1 and 2 are necessary and sufficient to certify performance in frequency intervals. The goal of this section is to provide synthesis conditions for the filtering problem. Note that even though several papers in the literature dealt with this problem [6], [12], [7], [8], the main contribution of this paper lies on the proposition of LMIs for the LF and MF filtering design problem using condition *iii*) of both Lemmas 1 and 2. In this view, this paper also aims to clarify the differences in terms of a bound for the $\mathscr{H}_\infty$ norm in LF and MF intervals using different extensions of the KYP Lemma, as exposed in the next two theorems.

Theorem *1:* Consider the linear time-invariant system in (1) and the pair of matrices (Φ_c, Ψ_1) that defines a low-frequency interval of the form $[0, \omega_\ell]$, $\omega_\ell > 0$. The following conditions are equivalent.

i) There exists a filter realization (A_f, B_f, C_f, D_f) such that $\|H(s)\|_\infty < \gamma, \forall s = j\omega, \omega \in [0, \omega_\ell]$.

ii) There exist hermitian matrices $Q_{11}, Q_{22}, P_{11}, P_{22} \in \mathbb{C}^{n\times n}$, matrices $Q_{12}, P_{12}, M_{11}, M_{21}, \hat{K}, M_{Af} \in \mathbb{C}^{n\times n}$, $M_{Bf} \in \mathbb{C}^{n\times q}$ such that inequality (9) holds.

iii) There exist matrices $F_{11}, F_{21}, \hat{K}, M_{Af} \in \mathbb{C}^{n\times n}$, $M_{Bf} \in \mathbb{C}^{n\times q}$ such that inequalities (10) hold for $i = 1, 2$ with $\tilde{\omega}_1 = \omega_\ell$ and $\tilde{\omega}_2 = -\omega_\ell$.

The filter realization that satisfies condition *i*) is obtained by $A_f = \hat{K}^{-1}M_{Af}$, $B_f = \hat{K}^{-1}M_{Bf}$, and C_f and D_f are variables of the optimization problem.

Proof: The relation *i*) ⇔ *ii*) can be demonstrated as follow. First, define matrices

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22}, \end{bmatrix}, \quad M = \begin{bmatrix} M_{11} & \hat{K} \\ M_{21} & \hat{K} \end{bmatrix},$$

and note that with these structures the products between matrix M and matrices A and B, whose expressions are presented in (4), are

$$M\mathsf{A} = \begin{bmatrix} M_{11}A + M_{Bf}C_y & M_{Af} \\ M_{21}A + M_{Bf}C_y & M_{Af} \end{bmatrix}, \quad M\mathsf{B} = \begin{bmatrix} M_{11}B_w + M_{Bf}D_{yw} \\ M_{21}B_w + M_{Bf}D_{yw} \end{bmatrix}.$$

Hence, inequality (9) can be rewritten as

$$\begin{bmatrix} -Q & P - M^* & 0 & 0 \\ \star & \omega_\ell^2 Q + \mathrm{He}(M\mathsf{A}) & M\mathsf{B} & \mathsf{C}^* \\ \star & \star & -\gamma^2 I & \mathsf{D}^* \\ \star & \star & \star & -I \end{bmatrix} \prec 0,$$

or, in view of Lemma 2, as

$$\begin{bmatrix} (\Phi_c \otimes P + \Psi_1 \otimes Q) & 0 & \begin{bmatrix} 0 \\ \mathsf{C}^T \end{bmatrix} \\ 0 & -\gamma^2 I & \mathsf{D}^T \\ \begin{bmatrix} 0 & \mathsf{C} \end{bmatrix} & \mathsf{D} & -I \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} 0 \\ M \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -I & \mathsf{A} & \mathsf{B} & 0 \end{bmatrix} \right\} \prec 0.$$

Then, the equivalence *i*) ⇔ *ii*) is established using Lemma 2 if and only if the following assertions hold true: *i)* the blocks that compose the second column of variable M can be made equal; *ii)* variables L and N can be eliminated without

$$
\begin{bmatrix}
-Q_{11} & -Q_{12} & P_{11}-M_{11}^* & P_{12}-M_{21}^* & 0 & 0\\
\star & -Q_{22} & P_{12}^*-\hat{K}^* & P_{22}-\hat{K}^* & 0 & 0\\
\star & \star & \omega_\ell^2 Q_{11}+\mathrm{He}(M_{11}A+M_{Bf}C_y) & \omega_\ell^2 Q_{12}+(M_{21}A+M_{Bf}C_y)^*+M_{Af} & M_{11}B_w+M_{Bf}D_{yw} & (C_z-D_fD_{yw})^*\\
\star & \star & \star & \omega_\ell^2 Q_{22}+\mathrm{He}(M_{Af}) & M_{21}B_w+M_{Bf}D_{yw} & -C_f^*\\
\star & \star & \star & \star & -\gamma^2 I & (D_{zw}-D_fD_{yw})^*\\
\star & \star & \star & \star & \star & -I
\end{bmatrix} \prec 0, \quad (9)
$$

$$
\begin{bmatrix}
\mathrm{He}(F_{11}A+M_{Bf}C_y-j\tilde{\omega}_iF_{11}) & M_{Af}+(F_{21}A+M_{Bf}C_y)^*-j\tilde{\omega}_i\hat{K}+j\tilde{\omega}_iF_{21}^* & F_{11}B_w+M_{Bf}D_{yw} & (C_z-D_fD_{yw})^*\\
\star & \mathrm{He}(M_{Af}-j\tilde{\omega}_i\hat{K}) & F_{21}B_w+M_{Bf}D_{yw} & -C_f^*\\
\star & \star & -\gamma^2 I & (D_{zw}-D_fD_{yw})^*\\
\star & \star & \star & -I
\end{bmatrix} \prec 0, \quad i=1,2. \quad (10)
$$

loss of generality. The former statement is proved using the arguments of paper [13], which shows that this constraint can be made without loss of generality. The latter is verified with a projection argument. In fact, variables L and N can be eliminated from the initial formulation if and only if the inequality

$$
Z^*\begin{bmatrix}(\Phi_c\otimes P+\Psi_1\otimes Q) & 0 & \begin{bmatrix}0\\ \mathsf{C}^T\end{bmatrix}\\ 0 & -\gamma^2 I & \mathsf{D}^T\\ \begin{bmatrix}0 & \mathsf{C}\end{bmatrix} & \mathsf{D} & -I\end{bmatrix}\underbrace{\begin{bmatrix}I & 0 & 0\\ 0 & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}}_{Z}\prec 0
$$

is satisfied; in other words, if and only if inequality

$$
\begin{bmatrix}-Q & 0 & 0\\ 0 & -\gamma^2 I & \mathsf{D}^*\\ 0 & \mathsf{D} & -I\end{bmatrix}\prec 0
$$

holds. Observe that this inequality is always valid with the assumption of Remark 1 and because matrix Q appears in the $(1,1)$ block in the left-hand side of inequality (9).

For the equivalence $iii) \Leftrightarrow i)$, a similar strategy is used. After applying Schur's complement with respect to the $(5,5)$ block in the matrix of inequality (10) and defining

$$
F=\begin{bmatrix}F_{11} & \hat{K}\\ F_{21} & \hat{K}\end{bmatrix},\quad F\mathsf{A}=\begin{bmatrix}F_{11}A+M_{Bf}C_y & M_{Af}\\ F_{21}A+M_{Bf}C_y & M_{Af}\end{bmatrix},
$$

$$
F\mathsf{B}=\begin{bmatrix}F_{11}B_w+M_{Bf}D_{yw}\\ F_{21}B_w+M_{Bf}D_{yw}\end{bmatrix},
$$

this inequality can be rewritten as

$$
\mathrm{He}\left\{\begin{bmatrix}F\\ 0\end{bmatrix}\begin{bmatrix}I & -j\tilde{\omega}_i I\end{bmatrix}(T\otimes I)\begin{bmatrix}\mathsf{A} & \mathsf{B}\\ I & 0\end{bmatrix}\right\} + \begin{bmatrix}\mathsf{C}^*\mathsf{C} & \mathsf{C}^*\mathsf{D}\\ \mathsf{D}^*\mathsf{C} & \mathsf{D}^*\mathsf{D}-\gamma^2 I\end{bmatrix}\prec 0,\quad i=1,2,
$$

where matrix T defines the homeomorphism between $\Lambda(\Phi_0,\Psi_0)$ and $\Lambda(\Phi_c,\Psi_1)$. For the low-frequency range, the coefficients of matrix Φ_0 are $\alpha=-1,\beta=0$, and $\gamma=\omega_\ell^2$, therefore $\mathscr{T}(s)=s$. In this way, as showed in condition $iii)$ of Lemma 1, $\tilde{\omega}_1=|\omega_\ell|$ and $\tilde{\omega}_2=-|\omega_\ell|$.

Therefore, the equivalence follows from analogous rationale used to establish $i) \Leftrightarrow ii)$, that is, proving that the variables that compose the second column of F can be made equal and that variable G can be eliminated, and then using the equivalence $i) \Leftrightarrow iii)$ of Lemma 1. ■

Theorem *2:* Consider the linear time-invariant system in (1) and the pair of matrices (Φ_c,Ψ_2) that defines a middle-frequency interval $[\omega_1,\omega_2]$. The following conditions are equivalent.

i) There exists a filter realization (A_f,B_f,C_f,D_f) such that $\|H(s)\|_\infty<\gamma,\forall s=j\omega,\omega\in[\omega_1,\omega_2]$.

ii) There exist hermitian matrices $Q_{11},Q_{22},P_{11},P_{22}\in\mathbb{C}^{n\times n}$, matrices $Q_{12},P_{12},M_{11},M_{21},\hat{K},M_{Af}\in\mathbb{C}^{n\times n}$, $M_{Bf}\in\mathbb{C}^{n\times q}$ such that inequality (11) holds, where

$$
\mathbf{R}_{34}=-\omega_1\omega_2Q_{12}+(M_{21}A+M_{Bf}C_y)^*+M_{Af}.
$$

iii) There exist matrices $F_{11},F_{21},\hat{K},M_{Af}\in\mathbb{C}^{n\times n}$, $M_{Bf}\in\mathbb{C}^{n\times q}$ such that inequality (10), with $\tilde{\omega}_1=\omega_1$ and $\tilde{\omega}_2=\omega_2$, hold.

The filter realization that satisfies condition $i)$ is obtained by $A_f=\hat{K}^{-1}M_{Af}$, $B_f=\hat{K}^{-1}M_{Bf}$, and C_f and D_f are variables of the optimization problem.

Proof: The proof is similar to the one of Theorem 1 and is omitted for brevity. ■

Even though Theorems 1 and 2 provide necessary and sufficient conditions for the filter design problem with LF or MF specifications, the matrices A_f obtained for the filter realization may not come out Hurwitz. To surpass this issue, an additional condition that assures asymptotic stability for the system in (2) is needed.

Suppose that inequality (9) holds. In this case, the extra condition

$$
\hat{K}+\hat{K}^*\succ 0, \quad (12)
$$

guarantees the desired stability of A_f, because (12) and the $(4,4)$ block in (9) imply the asymptotic stability of matrix A_f (Hurwitz matrix). To verify this claim, just substitute $M_{Af}=\hat{K}A_f$.

Similarly, observing the $(2,2)$ block of inequality (10), one can note that the restriction $\hat{K}=\hat{K}^*$ and inequality $\hat{K}\succ 0$ ensures asymptotic stability of A_f when using (10).

On the other hand, if inequality (11) is satisfied then asymptotic stability cannot be imposed by adding a constraint that uses only variables of the original problem. For this case, the following inequalities are used

$$
W\succ 0,\quad \begin{bmatrix}-\mathrm{He}(\hat{K}) & W+M_{Af}-\xi\hat{K}^*\\ \star & \xi\,\mathrm{He}(M_{Af})\end{bmatrix}\prec 0, \quad (13)
$$

$$\begin{bmatrix} -Q_{11} & -Q_{12} & P_{11}+j\omega_c Q_{11}-M_{11}^* & P_{12}+j\omega_c Q_{12}-M_{21}^* & 0 & 0 \\ \star & -Q_{22} & P_{12}^*+j\omega_c Q_{12}^*-\hat{K}^* & P_{22}+j\omega_c Q_{22}-\hat{K}^* & 0 & 0 \\ \star & \star & -\omega_1\omega_2 Q_{11}+\mathrm{He}(M_{11}A+M_{Bf}C_y) & \mathbf{R}_{34} & M_{11}B_w+M_{Bf}D_{yw} & (C_z-D_f D_{yw})^* \\ \star & \star & \star & -\omega_1\omega_2 Q_{22}+\mathrm{He}(M_{Af}) & M_{21}B_w+M_{Bf}D_{yw} & -C_f^* \\ \star & \star & \star & \star & -\gamma^2 I & (D_{zw}-D_f D_{yw})^* \\ \star & \star & \star & \star & \star & -I \end{bmatrix} \prec 0 \qquad (11)$$

where $0 \prec W = W^* \in \mathbb{R}^{n\times n}$ and $0 < \xi \in \mathbb{R}$ are additional variables[3].

Two important points are worth to be highlighted. First, note that even though the conditions of Theorems 1 and 2 solve the LF and MF filtering problem, this paper does not prove that the conditions remain necessary when an additional constraint for asymptotic stability is imposed. Second, complex matrices are obtained for the filter realization (which is non implementable).

The following propositions summarize the results presented in this paper, yielding sufficient conditions that provide asymptotically stable filter realizations with real parameters for the LF and MF filtering problems.

Proposition *1:* An asymptotically stable and real filter that assures $\|H(j\omega)\|_\infty < \gamma, \forall\omega \in [0,\omega_\ell]$ can be obtained by solving either

$$\begin{aligned} &\min_{\substack{Q_{11},Q_{12},Q_{22},P_{11},P_{12},P_{22},M_{11},\\ M_{21},\hat{K},M_{Af},M_{Bf},C_f,D_f}} \gamma^2 \\ &\textbf{s. t.} \quad (9),\ \hat{K}+\hat{K}^* \succ 0 \end{aligned} \qquad (14)$$

or

$$\begin{aligned} &\min_{F_{11},F_{21},\hat{K},M_{Af},M_{Bf},C_f,D_f} \gamma^2 \\ &\textbf{s. t.} \quad (10),\ \hat{K} \succ 0 \end{aligned} \qquad (15)$$

with $\tilde{\omega}_1 = -\omega_\ell$ and $\tilde{\omega}_2 = \omega_\ell$, where $\hat{K}, M_{Af} \in \mathbb{R}^{n\times n}$ and $M_{Bf} \in \mathbb{R}^{n\times q}$.

Proposition *2:* An asymptotically stable and real filter that assures $\|H(j\omega)\|_\infty < \gamma, \forall\omega \in [\omega_1,\omega_2]$ can be obtained by solving either

$$\begin{aligned} &\min_{\substack{Q_{11},Q_{12},Q_{22},P_{11},P_{12},P_{22},M_{11},\\ M_{21},\hat{K},M_{Af},M_{Bf},C_f,D_f}} \gamma^2 \\ &\textbf{s. t.} \quad (11),\ (13) \end{aligned} \qquad (16)$$

or (15) with $\tilde{\omega}_1 = \omega_1$ and $\tilde{\omega}_2 = \omega_2$, where $\hat{K}, M_{Af} \in \mathbb{R}^{n\times n}$ and $M_{Bf} \in \mathbb{R}^{n\times q}$.

As a final contribution of the paper, a numerical comparison between Propositions 1 and 2, which are based on different extensions for the KYP Lemma, is presented, including the treatment of uncertain systems in the polytopic form. The condition proposed in [7] is also investigated.

V. Numerical Examples

The routines were implemented in MATLAB , version 8.2.0.701 64 bits, using YALMIP [14] and Mosek [15]. The programming of the conditions of Propositions 1 and 2 is performed directly in YALMIP. However, the extension of these propositions to treat polytopic systems (discussed in Example 2) requires some explanations. First some decision variables are made polynomially parameter-dependent of fixed degree on the uncertain parameters and the LMIs, that are sufficient to check the positivity (or negativity) of the resulting polynomial inequalities, are obtained by means of Pólya's relaxations following the lines of [16]. Note that these trick tasks can be performed automatically using the package ROLMIP (Robust LMI Parser) [17].

[3]A search of the scalar ξ in the view of paper [8] could be employed to alleviate the conservatism when uncertain systems are investigated. Throughout this paper, the scalar ξ is fixed equal to 1.

Example 1. The purpose of this example is to compare the results of Propositions 1 and 2, and also [7], in terms of the accuracy of the $\mathscr{H}_\infty$ bounds and numerical complexity, which is inferred through the number of scalar variables V and LMI rows L. To set up this numerical experiment, 15 randomly generated systems given as in (1) with dimensions $n = 3,5$ and 8 (5 for each value of n) and $r = 2$, $p = 2$, and $q = 1$ are produced using the function `randn` of the MATLAB.

For each of these systems, if the generated matrix A is not Hurwitz, then the eigenvalues are displaced to the left by 1.1η, where η is the maximum value of the real part of the eigenvalues of A. The results obtained when the seed of the `randn` is 18 and $\omega_\ell = 20$ in Proposition 1, and $\omega_1 = 10$ and $\omega_2 = 100$ in Proposition 2 are the same, that is, approximately the same $\mathscr{H}_\infty$ bound was produced (within four decimal digits) for all the 15 systems considered. The LF and MF conditions presented in [7] were assessed as well, yielding similar results in terms of the $\mathscr{H}_\infty$ bound.

Figure 1 illustrates how the number of LMI rows (solid curves, left axis) and scalar variables (dashed lines, right axis) grows with the increase in the dimension of the system for the methods (14) (blue), (15) (green) and the continuous-time LF condition in [7] (red). Observe that, based on the results presented, the method (15) requires both less scalar variables and LMI rows. The behavior for MF is similar.

Example 2. Consider the robust asymptotically stable LTI system, treated in [18]

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & -1+0.3\alpha \\ 1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \\ y(t) &= \begin{bmatrix} -100+10\beta & 100 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(t) \end{aligned} \qquad (17)$$

where $|\alpha| \leq 3$ and $|\beta| \leq 1$. A polytopic model with four vertices can be obtained from this model. The conditions of Proposition 1 and 2 were programmed using ROLMIP by fixing all the optimization variables as polynomials of degree one except $\hat{K}, M_{Af}, M_{Bf}, C_f$ and D_f that were fixed as constant (degree zero).

As an example, for $\omega_\ell = 20$, Proposition 1 (adapted to cope with polytopic models) provided 7.2293 and 7.2295

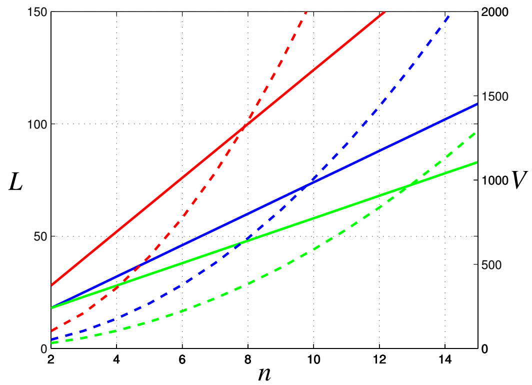

Fig. 1. Number of LMI rows – L, left axis, solid curves – and number of variables – V, right axis, dashed curves – for the two methods of Proposition 1 (blue - (14); green - (15)) and the LF condition of [7] (red) as a function of the number n of states of the system.

using (14) and (15), respectively, while the LF condition in [7] yields 7.6446. To investigate the behavior of the proposed conditions for the middle-frequency intervals and to compare them with the MF condition in [7], the value ω_1 is fixed as 0.3 and ω_2 varies from 0.35 to 30. These results are presented in Figure 2. Note that, for this example, Proposition 2 with (16) (blue) provided slightly lower bounds for small values of ω_2 when compared with method (15) (green), yielding similar results as ω_2 increases. The MF condition in [7] provided results that are comparable with Proposition 2, but using considerably more variables and LMI rows.

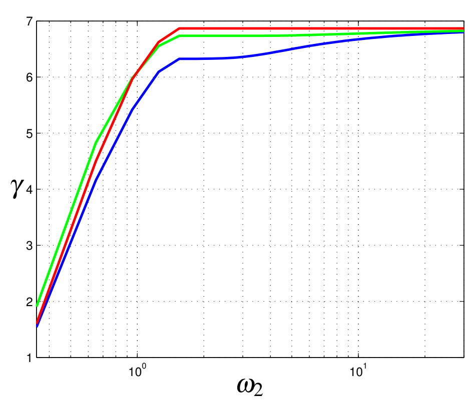

Fig. 2. $\mathscr{H}_\infty$ bounds provided by the robust filter applied to system (17) (Example 2) for the interval $[\omega_1 = 0.3, \omega_2]$ as a function of ω_2 using the two methods of Proposition 2 (green - (15); blue - (16)) and the MF condition from [7] (red).

VI. Conclusion

Synthesis conditions for the LF and MF filtering problems were proposed using two extensions of the KYP Lemma. Numerical examples were presented to evaluate the proposed conditions and to compare them with another condition of the literature in terms of the $\mathscr{H}_\infty$ bound and numerical complexity. From the numerical experiments, it seems that the two design conditions are equivalent (i.e., provide the same bounds to the $\mathscr{H}_\infty$ norm) but one has lower complexity (less LMI rows and variables) in the precisely known case. In the uncertain case, slightly smaller bounds have been obtained with the more complex proposition from this paper (even when compared to other conditions from the literature). The theoretical relationship between the two propositions in the general case remains a topic of future investigation.

References

[1] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma," *Syst. Control Lett.*, vol. 28, no. 1, pp. 7–10, June 1996.

[2] T. Iwasaki and S. Hara, "Generalized KYP lemma: Unified frequency domain inequalities with design applications," *IEEE Trans. Autom. Control*, vol. 50, no. 1, pp. 41–59, January 2005.

[3] G. Pipeleers and L. Vandenbergue, "Generalized KYP lemma with real data," *IEEE Trans. Autom. Control*, vol. 56, no. 12, pp. 2942–2946, December 2011.

[4] G. Pipeleers, T. Iwasaki, and S. Hara, "Generalizing the KYP lemma to multiple frequency intervals," *SIAM J. Control Optim.*, vol. 52, no. 6, pp. 3618–3638, 2014.

[5] M. R. Graham, M. C. de Oliveira, and R. A. de Callafon, "An alternative Kalman-Yakubovich-Popov lemma and some extensions," *Automatica*, vol. 45, no. 6, pp. 1489–1496, March 2009.

[6] T. Iwasaki and S. Hara, "Feedback control synthesis of multiple frequency domain specifications via generalized KYP lemma," *Int. J. Robust Nonlinear Control*, vol. 17, no. 5-6, pp. 415–434, March/April 2007.

[7] D. H. Lee, "An improved finite frequency approach to robust $\mathscr{H}_\infty$ filter design for LTI systems with polytopic uncertainties," *Int. J. Adapt. Control Signal Process.*, vol. 27, no. 11, pp. 944–956, 2013.

[8] L. B. R. R. Romão, M. C. de Oliveira, P. L. D. Peres, and R. C. L. F. Oliveira, "State-feedback and filtering problems using the generalized KYP lemma," in *Proc. 2016 IEEE Int. Symp. on Comput. Aided Control Syst. Des.*, Buenos Aires, Argentina, September 2016, pp. 1054–1059.

[9] T. Iwasaki, G. Meinsma, and M. Fu, "Generalized *S*-procedure and finite frequency KYP lemma," *Math. Probl. Eng.*, vol. 6, no. 2–3, pp. 305–320, 2000.

[10] R. E. Skelton, T. Iwasaki, and K. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*. Bristol, PA: Taylor & Francis, 1998.

[11] M. C. de Oliveira and R. E. Skelton, "Stability tests for constrained linear systems," in *Perspectives in Robust Control*, ser. Lecture Notes in Control and Information Science, S. O. Reza Moheimani, Ed. New York, NY: Springer-Verlag, 2001, vol. 268, pp. 241–257.

[12] H. Gao and X. Li, "$\mathscr{H}_\infty$ filtering for discrete-time state-delayed systems with finite frequency specifications," *IEEE Trans. Autom. Control*, vol. 56, no. 12, pp. 2935–2941, December 2011.

[13] L. Frezzatto, M. C. de Oliveira, R. C. L. F. Oliveira, and P. L. D. Peres, "Robust H_∞ filter design with past output measurements for uncertain discrete-time systems," *Automatica*, vol. 71, pp. 151–158, September 2016.

[14] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proc. 2004 IEEE Int. Symp. on Comput. Aided Control Syst. Des.*, Taipei, Taiwan, September 2004, pp. 284–289.

[15] MOSEK ApS, *The MOSEK optimization software*, 2015, http://www.mosek.com.

[16] R. C. L. F. Oliveira and P. L. D. Peres, "Parameter-dependent LMIs in robust analysis: Characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations," *IEEE Trans. Autom. Control*, vol. 52, no. 7, pp. 1334–1340, July 2007.

[17] C. M. Agulhari, R. C. L. F. Oliveira, and P. L. D. Peres, "Robust LMI parser: A computational package to construct LMI conditions for uncertain systems," in *XIX CBA*, Campina Grande, PB, Brazil, September 2012, pp. 2298–2305.

[18] M. J. Lacerda, R. C. L. F. Oliveira, and P. L. D. Peres, "Robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filter design for uncertain linear systems via LMIs and polynomial matrices," *Signal Process.*, vol. 91, no. 5, pp. 1115–1122, May 2011.