Scalable and data-driven approaches to convex programming

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Abstract

Mathematical optimisation plays a crucial role in providing efficient solutions to modern engineering applications. Despite enormous advances in the past decades, some of these applications involve solving optimisation problems that may not be computationally tractable or that call for new theoretical advancements. Hence, obtaining robust and scalable methods and developing new analytic tools to optimisation problems is of paramount importance. This thesis addresses some of the main challenges when solving optimisation programs, namely, scalability, the presence of integer decision variables and the presence of uncertainty.

Scalability throughout this thesis is achieved by means of distributed computation. Technological advancements that lead to the increase of computational devices per capita have attracted research on the development of algorithms that can exploit such a distributed and unstructured computational power to solve large-scale optimisation problems. In this context, multi-agent optimisation has emerged, as it enables distributed computation by allowing devices (or agents) to communicate over a network. This thesis proposes an algorithmic scheme based on subgradient averaging to perform multi-agent optimisation.

Some optimisation problems are computationally intractable due to the presence of integer variables. Optimising over integers is, in general, hard due to the lack of polynomial-time algorithms to obtain an optimal solution. In fact, some well-known NP-hard problems can be cast as optimisation programs involving integer decision variables. In this thesis we investigate the so-called actuator placement problem and show that a particular instance of this problem, despite being an integer program can be solved exactly by means of a convex relaxation. We also link such a relaxation to the multi-agent optimisation framework explored previously, showing how distributed schemes can be leveraged to obtain the optimal solution.

Other challenging optimisation problems arise with the presence of uncertain constraints. Indeed, even the well-studied class of linear optimisation problems may require theoretical and algorithmic advancements under this type of constraints. Uncertain constraints represent our lack of knowledge about the underlying phenomena and appear in several applications, including the optimal power flow problem under renewable energy generation, robotics, and transportation applications. Motivated by this fact, the last part of this thesis focuses on a randomised approximation to chance-constrained optimisation problems. We show a new theoretical bound on the probability of constraint violation that can improve the conservatism with respect to the state-of-the-art.

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This thesis is dedicated to
my dear wife Isabela Gomes
for her incredible patience and support throughout this journey.

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Abstract

Mathematical optimisation plays a crucial role in providing efficient solutions to modern engineering applications. Despite enormous advances in the past decades, some of these applications involve solving optimisation problems that may not be computationally tractable or that call for new theoretical advancements. Hence, obtaining robust and scalable methods and developing new analytic tools to optimisation problems is of paramount importance. This thesis addresses some of the main challenges when solving optimisation programs, namely, scalability, the presence of integer decision variables and the presence of uncertainty.

Scalability throughout this thesis is achieved by means of distributed computation. Technological advancements that lead to the increase of computational devices per capita have attracted research on the development of algorithms that can exploit such a distributed and unstructured computational power to solve large-scale optimisation problems. In this context, multi-agent optimisation has emerged, as it enables distributed computation by allowing devices (or agents) to communicate over a network. This thesis proposes an algorithmic scheme based on subgradient averaging to perform multi-agent optimisation.

Some optimisation problems are computationally intractable due to the presence of integer variables. Optimising over integers is, in general, hard due to the lack of polynomial-time algorithms to obtain an optimal solution. In fact, some well-known NP-hard problems can be cast as optimisation programs involving integer decision variables. In this thesis we investigate the so-called actuator placement problem and show that a particular instance of this problem, despite being an integer program can be solved exactly by means of a convex relaxation. We also link such a relaxation to the multi-agent optimisation framework explored previously, showing how distributed schemes can be leveraged to obtain the optimal solution.

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to chance-constrained optimisation problems. We show a new theoretical bound on the probability of constraint violation that can improve the conservatism with respect to the state-of-the-art.

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Notation

$\mathbb{R}$	Denotes the real numbers
$\mathbb{R}^n$	Denotes the cartesian product $\prod_{i=1}^{n} \mathbb{R}$
$\mathbb{Z}$	Denotes the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{N}$	Denotes the natural numbers $\{1, 2, \ldots\}$
I	Denotes an arbitrary set of index
$(x_n)_{n\in\mathbb{N}}$	Denotes a sequence $\{x_1, x_2, \ldots\}$
2^X	Denotes the power set of a set X
$\text{int}(A)$	Denotes the interior of a set A
$\text{cl}(A)$	Denotes the closure of a set A
$\text{conv}(A)$	Denotes the convex hull of a set A
A^c	Denotes the complement of a set A
$A \subset B$	Denotes that every element of A is an element of B
$(X, \mathcal{T}_X)$	Denotes a topological space
$(\Delta, \mathcal{F})$	Denotes a measurable space
$(\Delta, \mathcal{F}, \mathbb{P})$	Denotes a probability measure space
$\sigma(\mathcal{T})$	Denotes the σ-algebra generated by the collection of subsets given by $\mathcal{T}$
$\mathbb{E}\{Z\}$	Denotes the expectation of a random variable Z
$f^\star$	Denotes the optimal solution of an optimisation problem
$\partial f(x)$	Denotes the subdifferential of f at a point x
$\mathcal{A}(\cdot)$	Denotes a mapping $\mathcal{A} : \Delta^m \to 2^\Delta$ in the compression learning framework

1
Introduction

1.1 Motivation

Modern engineering methods require solving optimisation problems (obtained, for instance, by means of optimal control theory) that strike a balance between performance and robustness against disturbances and unmodelled dynamics. These problems offer flexibility to the designer, as they allow to tailor the objective function to meet performance specifications and the constraints to guarantee a robust and safe solution. In fact, this strategy has led to several developments in many different areas of engineering and related fields, including control theory [20], [107], [126], [132], power systems [36], [71], [77], [148], statistics [1], [47], [52], [88], and finance [37], [43], [80], [115]. However, while we have experienced several advances in past decades, especially the development of polynomial-time algorithms [3], [106], [144] to solve the broad and important class of convex optimisation problems, there is a plethora of large-scale engineering applications that cannot be addressed under the current technology. Hence, it is of paramount importance to study limitations of current optimisation schemes.

As an example, consider the design of controllers using the H_2 and H_∞ norms for linear time-invariant (LTI) systems. Well-known formulations of these problems require the solution of semi-definite programs (SDPs) [8], [20], [59], [97], [108], [132] and extensions to cope with uncertain LTI models whose matrices belong to a polytope of known vertices are available [69], [107], [126]. However, H_2 and H_∞ control problems problems may become difficult to solve if the dimension of the state-space model is too large, or if the uncertainty affecting the system has a more complicated description. Thus we need more tools and ideas to perform a robust and safe design using mathematical optimisation. Motivated by this discussion, this thesis focuses on overcoming the intractability of optimisation-based methods from three different perspectives: scalability, the presence of integer decision variables and the presence of uncertain constraints.

Scalability within an optimisation problem admits several interpretations. The complexity of optimisation algorithms is usually characterised by means of the required computation and storage to run the algorithm in terms of the input size. For instance, this complexity for standard interior-point algorithms is known to be polynomial in n, where n is the number of optimisation variables [106]. As a rough approximation, it increases more than 10^3 times when the number of variables is increased tenfold. Hence, interior point methods cannot be applied to large-scale optimisation problems. Recent progress towards scalability has been made by exploring the structure of SDPs [58], [96], [143], [154] using chordal sparsity, which can then reduce the computation to be polynomial in r, where r is the size of the maximum clique associated with the sparsity pattern of the problem data. This can lead to a significant speed-up if r is much less than n.

In this thesis, we approach scalability from a different perspective. Our goal is to address large-scale problems by leveraging distributed computation [11], [92], [101], [103], [123]. Under this setting, the optimisation problem is solved using a collection of processors, also reffered to as agents, that holds private information (e.g., part of the objective function and local constraint sets) and that is not willing to share such an information with a single processing unit. We allow agents to comminucate with their neighbours by means of a network and to update their local estimates of the optimal solution based on the received information. The main challenge involves proving that all generated estimates converge to the optimal solution of the underlying (global) optimisation problem. Scalability here is then related to the rate of convergence of these iterates and to the influence of the connectivity of the communication network in such convergence.

Intractability can also arise in the presence of integer variables. Indeed, many NP-hard problems [18], [41], [72] possess an equivalent formulation involving an optimisation over integer variables, e.g., the actuator placement problem. Steering the states of large-scale systems can be prohibitively expensive if one places an actuator at each state of the network; hence, it is usually necessary to limit the number of actuators to a subset of these states. To this end, given a fixed number of actuators, the goal of the actuator placement problem is to select the optimal – according to some metric – position of these actuators. Depending on the metric employed, we obtain different formulations of the problem. Paper [114] studied, for instance, how to place actuators and sensors so that the resulting system is structurally controllable and observable (see [83], [134]), while [109] establishes NP-hardeness of finding a smallest set of actuators that make the resulting system controllable. This negative statement, however, does not preclude particular instances to be tractable, and more research is needed to identify and propose efficient algorithmic schemes for such instances. Approximation algorithms for this class of problems have been recently addressed in [40], [63], [110].

The third challenge is the presence of uncertain constraints. Even for the well-studied class of linear programs, which is under development since at least the 50s [70], [151], providing guarantees to the optimal solution is elusive under the presence of uncertainty – progress has recently been made under restrictive scenarios [6], [7], [14], [15], but no general solution is known. Throughout this thesis, we treat uncertainty in the constraints by means of a chance-constrained formulation to optimisation problems. Such formulation allows some level of constraint violation, thus distinguishing from the robust paradigm, which requires the optimal solution to satisfy the constraints for all realizations of the uncertain parameters. Unfortunately, the chance-constrained formulation is only tractable for some specific probability distributions [105], [118] and its general formulation may lead to non-convex problems, even when all the functions defining the constraints are convex.

A well-known approximation to chance-constrained optimisation problems is the scenario approach theory [29], [31], [32], [87], which relies on a set of independent and identically distributed samples either from a data set or from a probabilistic model of the uncertainty. The main idea of the scenario approach theory is to construct an auxiliary convex problem, called a scenario program, and study the feasibility guarantees associated to its optimal solution, showing that a feasible solution to the original chance-constrained problem can be produced with high-probability. The scenario approach theory does not impose any assumption on the underlying distribution if the uncertainty's distribution is fixed but unknown. However, as a price for such a general result, the associated optimal objective value can be conservative. To this end, the theory has been extended in [22], [30] to trade feasibility to performance by allowing a certain fraction of the scenarios to be discarded. As opposed to the main result in [29] that is tight for a certain class of scenario programs, the result of [30] is not known to be tight. In this thesis we provide a tight characterization for the probability of constraint satisfaction for such problems, thus improving upon the existing results in [22], [30].

The aforementioned limitations of optimisation-based techniques complicate their use to large scale engineering applications. This thesis provides a step towards overcoming these limitations by dealing with each of them seperately and offering new theoretical results.

1.2 Thesis outline and contributions

The main contributions of this thesis are new theoretical and algorithmic results related to the limiting factors described in Section 1.1. Below we provide a detailed description of each chapter, highlighting its main contributions.

- In Chapter 2, we introduce some mathematical background used throughout the thesis. We start with some basic topological concepts important to some technical results of Chapter 3 and to the understanding of the measure-theoretic framework used in Chapters 5 and 6. These concepts include the abstract notion of a topological space, and the definition of interior and closure of subsets of a topological space. We then introduce the main concepts related to measure theory that will be employed in subsequent chapters, which include the definition of σ-algebra and probability measure spaces. Convex functions and convex optimisation play a crucial role in this thesis, and these are also introduced in this chapter. We show existence of a subgradient for convex functions in the interior of their domain, an important property that is explored in Chapter 3. We also review duality theory, as this will be used in Chapter 4. Finally, we also prove a fundamental result on compression learning that are used in Chapters 5 and 6.

- In Chapter 3, we study multi-agent optimisation as a way to achieve scalability. Our setup considers a separable objective function and assumes that agents communicate over a network to solve an underlying optimisation problem. We propose a new distributed scheme based on subgradient averaging that consists of a state-averaging step, where the current local estimate is shared with neighbouring agents, a subgradient-averaging step, where a subgradient of the local function evaluated at the average computed in the previous step is used to compose a proxy for the subgradient of the global function, and a local update, in which the current averaged estimate is projected onto the local sets. The distinctive feature of our proposed scheme with respect to other similar algorithms in the literature is that its analysis holds simultaneously for time-varying networks, different constraint sets per agent, and subgradient averaging, features that have so far been considered separately in the literature. We also establish the rate at which the generated sequences converge to the optimal set of the global problem. This rate recovers standard results for the centralised problem under similar assumptions. This chapter is based on the papers

  - "*Convergence rate analysis of a subgradient averaging algorithm for distributed optimisation with different constraint sets*", L. Romao, K. Margellos, G. Notarstefano, and A. Papachristodoulou. Proceedings of the 58th Conference on Decision and Control, pp. 7448–7453, 2019.
  - "*Subgradient averaging for multi-agent optimisation with different constraint sets*", L. Romao, K. Margellos, G. Notarstefano, and A. Papachristodoulou. vol. 131. Automatica. 2021

- In Chapter 4, we investigate an optimisation problem involving integer variables derived from a formulation of the actuator placement problem. Given a network and a set of actuators, our task is to place these actuators in order to maximise the trace of the controllability Gramian. Using properties of integral polyhedra, we show through a sequence of reformulations that the optimal solution of this problem can be determined by means of a linear program without introducing any relaxation gap. This allows us to obtain the optimal solution using a primal-dual distributed algorithm, thus providing a scalable approach to the actuator placement. We illustrate the main features of our approach by means of a case study involving a simplified model of the European power grid. This chapter is based on

  - "Distributed Actuator Selection: Achieving Optimality via a Primal-Dual Algorithm", L. Romao, K. Margellos, and A. Papachristodoulou. Accepted for publication in the IEEE Control Systems Letters, vol. 2, no. 4, pp. 779–784, 2018.

- In Chapter 5, we study optimisation problems with uncertain constraints under the lens of the scenario approach theory. We are interested in trading feasibility to performance, hence focus on scenario programs with discarded constraints. We revisit the so-called sampling and discarding approach that provides feasibility guarantees for the scenario solution when the decision maker is allowed to discard some of the original scenarios. We analyse a scheme that consists of a cascade of optimization problems, where at each step we remove a superset of the active constraints. We leverage results from the compression learning literature to produce a tighter bound for the probability of constraint violation of the obtained solution compared to the existing state-of-the-art when scenarios are removed in multiples of the dimension of the optimisation problem. We also show that the proposed bound is tight by describing a class of scenario problems that achieves the given upper bound. The proposed methodology is compared to a discarding scheme based on a greedy removal strategy. This chapter is based on

  - "On the exact feasibility of convex scenario programs with discarded constraints", L. Romao, K. Margellos, and A. Papachristodoulou. 2021. Submitted to IEEE Transactions on Automatic Control.
  - "Tight generalization guarantees for the sampling and discarding approach to scenario optimization", L. Romao, K. Margellos, and A. Papachristodoulou. Proceedings of the 59th IEEE Conference on Decision and Control, pp. 2228—2233, 2020.

- In Chapter 6, we extend the analysis of Chapter 5 to the case of an arbitrary number of removed scenarios. There are two main messages with the results in this chapter. The first one states that the feasibility guarantees one can offer to the resulting solution are better than the standard sampling-and-discarding bound applied to an arbitrary number of discarded scenarios. The second, and perhaps surprising, one highlights that, unless we impose a restrictive assumption on the class of scenario programs, the obtained bound for an arbitrary number of removed samples is slightly more conservative. This is due to the fact that scenarios in the support set of the intermediate stages can be disjoint. We also provide a result valid for min-max scenario programs, where at the last stage we improve the cost by moving in the direction of the epigraphic variable. This chapter is based on

- “*Tight sampling and discarding bounds for scenario programs with an arbitrary number of removed samples*”, L. Romao, K. Margellos, and A. Papachristodoulou, 2021. To appear in the proceedings of the 3rd Annual Learning for Dynamics and Control Conference. Zurich, Switzerland.
- “*Scenario programs with discarded scenarios: feasibility bounds for an arbitrary number of removed scenarios*,”L. Romao, K. Margellos, and A. Papachristodoulou, 2021. Submitted to Automatica.

- Chapter 7 summarises the thesis’ contributions and discusses directions for future work.

2
Mathematical preliminaries

In this chapter, we briefly introduce the main mathematical concepts that will be used in the thesis. This chapter aims to keep this manuscript self-contained and is structured as follows. In Sections 2.1 and 2.2, we review some topological and measure-theoretic concepts that form the foundation to well-known concepts in convex analysis and probability. In Section 2.3, we give a brief introduction to convex functions and convex optimisation problems. In Section 2.4, we review some learning-theoretic results that constitute the foundation of Chapters 5 and 6. In Section 2.5, a list of further reading on the material presented in this chapter is given. Finally, in Section 2.6 we provide a diagram that illustrates how the sections in this chapter relate to the main results of this thesis.

2.1 Basic topology

2.1.1 Topological spaces

We start with a brief introduction to some topological concepts that are used throughout this thesis. This treatment is by no means complete and is aimed at introducing the reader to the minimal knowledge necessary to understand some measure-theoretic concepts presented later. In doing so, we also aim to enhance our understanding on some concepts that are topological in nature, such as closure, interior, convex hulls, continuity, among others. To this end, we start with a definition of topological spaces.

Definition 1. *Let X be a non-empty set. We say that a collection of subsets of X, denoted by $\mathcal{T}$, is a topology for X if the following conditions hold:*

i) $X, \emptyset \in \mathcal{T}$;

ii) For all $U, V \in \mathcal{T}$, *we have that* $U \cap V \in \mathcal{T}$;

iii) Let I be an arbitrary index set. Then if $X_i \in \mathcal{T}$ for all $i \in I$, we have that $\cup_{i\in I} X_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space.

The subsets $U \in \mathcal{T}$ are called open sets. Note that $\mathcal{T}$ is closed under finite intersection and arbitrary union. There is no particular reason for defining which subsets of X are open; in fact, to construct a topological space we only need to construct a collection of subsets satisfying Definition 1.

Once we have a topological space $(X, \mathcal{T})$, we can define the collection of closed subsets of X. If[1] $B \subset X$ and its complement is an element of $\mathcal{T}$, then B is called a closed subset. It is a fact that the collection of closed subsets is closed under finite union and arbitrary intersection.

Throughout this thesis, we are interested in topological spaces whose open sets (i.e., its topology) are induced by a norm. To this end, let us introduce the notion of normed spaces.

Definition 2. *A pair $(X, \|\cdot\|)$ is called a normed space if X is a vector space and the function $\|\cdot\| : X \to \mathbb{R}$ satisfies*

i) For all $x \in X$, $\|x\| = 0$ *implies that* $x = 0$.

ii) For all $\alpha \in \mathbb{R}$ *and for all* $x \in X$, *we have* $\|\alpha x\| = |\alpha|\|x\|$.

[1] Let A and B be subsets of a set X. Throughout this thesis we will be using the notation $A \subset B$ whenever A is contained in B, i.e., whenever every element of A is an element of B. In the literature, it is common to use $A \subseteq B$ for the above meaning, and reserve the notation $A \subset B$ for strict inclusion. With our notation, strict inclusion can be represented as $A \subset B$ and $B \not\subset A$.

iii) For all $x, y \in X$, we have $\|x+y\| \leq \|x\| + \|y\|$.

In other words, normed vector spaces possess a induced distance function given by $d(x, y) = \|x - y\|$, as a consequence of Definition 2. Given a normed space $(X, \|\cdot\|)$, we can construct a topology to X by means of the open balls $\mathbb{B}(x, r) = \{y \in X : d(x, y) < r\}$, where $x \in X$ and $r > 0$. Throughout this thesis, we think of $\mathbb{R}^n$ as a topological space with topology induced by the standard Euclidean distance $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$.

We now define the notions of continuity and compactness in topological spaces that will be employed in all subsequent chapters of this thesis.

Definition 3. *Let $(X, \mathcal{T})$ be a topological space. A subset $K \subset X$ is said to be compact if for all collection of open sets $(U_i)_{i\in I} \subset \mathcal{T}$ such that $K \subset \cup_{i\in I} U_i$, there exist a finite set of indices $\{1, \ldots, \ell\} \subset I$ such that $K \subset \cup_{i=1}^{\ell} U_i$.*

The collection of subsets $(U_i)_{i\in I}$ is called an open cover of K. Using this terminology, Definition 3 states that K is compact if for all open cover there is a finite subcover. A theorem due to Heine-Borel [128, Theorem 2.41] states that $K \subset \mathbb{R}^n$ (i.e., if we consider $X = \mathbb{R}^n$ and $\mathcal{T}$ being the topology induced by the Euclidean metric) is compact if and only if K is closed and bounded.

Definition 4. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Then a function $f : X \to Y$ is continuous if for all open sets U of $\mathcal{T}_Y$ we have that $f^{-1}(U)$ is open in X, i.e., for all $U \in \mathcal{T}_Y$ we have that*

$$f^{-1}(U) = \{x \in X : f(x) \in U\} \in \mathcal{T}_X.$$

Definition 4 gives a notion of continuity between arbitrary topological spaces that is independent of any metric. It is possible to check that this notion coincides with the usual ϵ-δ argument when restricted to normed spaces [99, Chapter 2, Exercise 1]. There is an interesting connection between continuity and compactness, as presented in the sequel.

Proposition 1 ([99], Chapter 5, Theorem 5.5). *Let $f : X \to Y$ be a continuous function and $K \subset X$ be a compact set. Then the set $f(K) = \{y \in Y : y = f(x),\ \text{for } x \in K\}$ is compact.*

Proof. To show that $f(K)$ is compact we use Definition 3. Let $(U_i)_{i\in I}$ be an open cover of $f(K)$, our goal is to show that there is a finite subcover. Since $f(K) \subset \cup_{i\in I} U_i$ and f is continuous, we have that $K \subset \cup_{i\in I} f^{-1}(U_i)$, i.e., $(f^{-1}(U_i))_{i\in I}$ is an open cover for K. Then, by compactness of K, there exists a finite subcover, i.e., there exist indices $\{1, \ldots, \ell\} \subset I$ such that $K \subset \cup_{i=1}^{\ell} f^{-1}(U_i)$. However, this implies that $f(K) \subset \cup_{i=1}^{\ell} U_i$, thus showing that $f(K)$ is compact. This concludes the proof of Proposition 1. □

2.1.2 Interior, relative interior, closure, convex sets, convex hull, and affine hull of a subset

Let $(X, \mathcal{T})$ be a topological space (e.g., $\mathbb{R}^n$ with the topology induced by the Euclidean metric) and $B \subset X$ be given. Two subsets constructed from B are of interest. The first one is the interior of B, formally defined next.

Definition 5. *Let $B \subset X$ and define the collection of subsets*

$$\mathcal{B} = \{A \subset X : A \in \mathcal{T}, A \subset B\}.$$

The interior of B, denoted by $\text{int}(B)$, is given by $\text{int}(B) = \cup_{A\in\mathcal{B}} A$.

By definition of a topology (more specifically, item *iii*) in Definition 1), we have that $\text{int}(B)$ is an open set. Besides, it is also the largest open set contained in B, as any open set contained in B is in the collection $\mathcal{B}$ in Definition 5. The other important subset constructed from B is called the closure of B.

Definition 6. *Let $B \subset X$ and define the collection of subsets*

$$\mathcal{B} = \{A \subset X : A^c \in \mathcal{T}, B \subset A\},$$

where A^c is the complement of A in X, i.e., $A^c = X \setminus A$. The closure of B, denoted by $\text{cl}(B)$, is given by $\text{cl}(B) = \cap_{A\in\mathcal{B}} A$.

Similar as before, we can observe that $\mathrm{cl}(B)$ is the smallest closed set containing B. Besides, by Definitions 5 and 6, we have that $B = \mathrm{int}(B)$ if B is open, and $B = \mathrm{cl}(B)$ if B is closed. In general, we have that $\mathrm{int}(B) \subset B \subset \mathrm{cl}(B)$.

Let $(X, \mathcal{T})$ be a vector space [2] and $Y \subset X$ be a subspace of X. The subpace Y can be equiped with a topology induced by the topology of the ambient space X. To this end, we define

$$\mathcal{T}_Y := \{A \cap Y : A \in \mathcal{T}\}, \tag{2.1}$$

and let the open sets of Y be the elements of $\mathcal{T}_Y$. It is trivial to check that $(Y, \mathcal{T}_Y)$ satisfies the properties of Definition 1, so it is a well-defined topological space. Note that the elements of $\mathcal{T}_Y$ are obtained by taking the intersection of an open set of X with Y. The topology $\mathcal{T}_Y$ in (2.1) is called the subspace topology (please refer to [99] for more details). Given any subset $B \subset X$ (not necessarily a subspace), there is a natural way to produce a subspace of X.

Definition 7. *Let $B \subset X$, where X is a vector space, and consider the collection of subspaces*

$$\mathcal{B} = \{Y : Y \text{ is a subspace of } X, B \subset Y\}.$$

The affine hull of B, denoted by $\mathrm{aff}(B)$, is given by $\cap_{Y \in \mathcal{B}} Y$.

In other words, the affine hull of a subset $B \subset X$ is the smallest subspace that contains B. This construction leads naturally to the definition of the relative interior of a subset.

Definition 8. *Let $B \subset X$. The relative interior of B, denoted by $\mathrm{ri}(B)$, is defined as the interior of B with respect to the subspace topology of $\mathrm{aff}(B)$, i.e.,*

$$\mathrm{ri}(B) = \cap_{B \in \mathcal{B}} B,$$

[2] A vector space consists of a set X and a field $\mathbb{F}$, which is usually either $\mathbb{R}$ or $\mathbb{C}$, for which addition between elements of X and multiplication between an element of $\mathbb{F}$ and an element of X are well-defined and well-behaved (i.e., they are associative, distributive, etc). When these two operations are continuous with respect to the topology of X, a beautiful mathematical structure called topological vector spaces emerges (see [129] for more details). In this chapter, whenever we refer to a topological space $(X, \mathcal{T})$ being a vector space, we are implicity invoking this structure of topological vector spaces. A typical example of a topological vector spaces is, for instance, $\mathbb{R}^n$ equiped with the topology induced by the Euclidean distance.

where $\mathcal{B} = \{A \subset X : A \in \mathcal{T}_{\text{aff}(B)}, A \subset B\}$.

To construct the relative interior of B we proceed intuitively as follows: (1) we compute the affine hull of B as in Definition 7; (2) we then equip $\mathcal{T}_{\text{aff}(B)}$ with the subspace topology as in (2.1); (3) we compute the interior of B with respect to $\mathcal{T}_{\text{aff}(B)}$.

A natural question one may ask is how the interior and relative interior of a set B are related. First, since $\text{int}(B)$ is the largest open set (with respect to the topology of X) contained in B, we have that $\text{int}(B) = \text{int}(B) \cap \text{aff}(B) \in \mathcal{T}_{\text{aff}(B)}$, where the equality holds due to the fact that $\text{int}(B)$ is contained in B and the fact that $\text{int}(B) \cap \text{aff}(B) \in \mathcal{T}_{\text{aff}(B)}$ by the definition of the subspace topology. We can then conclude that $\text{int}(B) \subset \text{ri}(B)$. Moreover, the previous inclusion can be strict. To check this, consider $\mathbb{R}^2$ with the topology given by the Euclidean norm and let B be any line segment. As the affine hull of B has dimension one, the interior of B is empty; however, its relative interior is non-empty as there exists an open set of $\mathbb{R}^2$ that has non-empty intersection with B. Let us now focus our attention to a special collection of subsets of X.

Definition 9. *Let X be a vector space. We say that a set $C \subset X$ is convex if for all $x, y \in C$ and $\theta \in (0, 1)$ we have that $\theta x + (1 - \theta)y \in C$.*

The collection of all convex sets of a vector space is closed under arbitrary intersection, i.e., $\cap_{i \in I} X_i$ is convex whenever all the sets X_i, $i \in I$, are convex. However, the union of two convex sets is not necessarily convex, e.g., consider two disjoint intervals in the real line and take their union. There is a convex set associated to any subset of a vector space.

Definition 10. *Let $C \subset X$, where X is a vector space, and consider the collection of subsets*

$$\mathcal{C} = \{A \subset X : C \subset A, A \text{ is convex}\}.$$

Then $\text{conv}(C) = \cap_{A \in \mathcal{C}} A$ *is the convex hull of the set C.*

One can check that conv(C) is the smallest convex set containing C. In fact, since the collection of convex sets is closed under intersection we have that conv(C) is convex and if A is convex and $C \subset A$, then we have by definition that such an A is in $\mathcal{C}$, so we may conclude that conv$(C) \subset A$. Definition 10, however, is abstract and may not be useful for computations; hence, the following proposition provides an equivalent definition of the convex hull of a set.

Proposition 2 ([122], Section 1)**.** *Let $C \subset X$, where X is a vector space. We have that*

$$\operatorname{conv}(C) = \left\{ \sum_{i=1}^{\ell} \theta_i x_i : \sum_{i=1}^{\ell} \theta_i = 1, \theta_i \geq 0, x_i \in C \text{ for all } i = 1, \dots, \ell, \text{ and } \ell \in \mathbb{N} \right\}.$$

An alternative proof of Proposition 2 can be found in [138]. It states that the convex hull of a set can be found by forming all finite convex combinations of elements of C. This is a general result and holds true even if the underlying space is infinite-dimensional. However, if X is finite-dimensional (e.g., $X = \mathbb{R}^n$), then we can refine the above theorem by restricting $\ell \leq n+1$ and this consists of the well-known Caratheodory's theorem [122, Section 17].

2.2 Measure theory

The last two chapters of this thesis deal with uncertain optimisation problems, so in this section we aim to provide a brief review on some measure-theoretic concepts that will be essential in those chapters. The concepts introduced in Section 2.1 play an important role in this section.

Definition 11. *Let Δ be a non-empty set and $\mathcal{F}$ be a collection of subsets of Δ satisfying*

i) $\Delta \in \mathcal{F}$.

ii) For all $F \in \mathcal{F}$, we have that F^c also belongs to $\mathcal{F}$.

iii) For all $F_i \in \mathcal{F}, i \in \mathbb{N}$, we have that $\cup_{i\in\mathbb{N}} F_i \in \mathcal{F}$.

The collection of subsets $\mathcal{F}$ is called σ-algebra and the pair $(\Delta, \mathcal{F})$ is called a measurable space.

It follows that the σ-algebra is closed under complement and countable union and intersection (the latter is due to items *ii)* and *iii)* in the above definition). The elements of the σ-algebra $\mathcal{F}$ can be thought as the collection of possible events in a probabilistic context. One can also show that the intersection of an arbitrary collection of σ-algebras is also a σ-algebra and, being so, given a collection of subsets of Δ the smallest σ-algebra containing such a collection is well-defined. For instance, let $\mathcal{T}$ be a collection of subsets of Δ. We can define the smallest σ-algebra containing $\mathcal{T}$ as

$$\sigma(\mathcal{T}) = \bigcap \{\mathcal{F} : \mathcal{T} \subset \mathcal{F}, \mathcal{F} \text{ is a } \sigma\text{-algebra}\}.$$

Common measurable spaces arise from topological ones. Indeed, let $(X, \mathcal{T})$ be a topological space and consider the σ-algebra generated by the collection of open sets $\mathcal{T}$ (in this case $\Delta = X$). The resulting σ-algebra (i.e., $\mathcal{F} = \sigma(\mathcal{T})$) is called the Borel σ-algebra. When the topology of X is clear from the context we may denote its Borel σ-algebra by $\mathcal{B}(X)$. For instance, the Borel σ-algebra of $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$, is the σ-algebra generated by the topology[3] associated to the standard metric in $\mathbb{R}$. More generally, the Borel σ-algebra in $\mathbb{R}^n$ is the σ-algebra generated by the rectangles, i.e.,

$$\mathcal{B}(\mathbb{R}^n) = \sigma\left(\left\{\prod_{i=1}^{n}(a_i, b_i) : a_i < b_i \text{ for all } i = 1, \dots, n\right\}\right).$$

In this thesis we only consider $\mathbb{R}$ and $\mathbb{R}^n$ equipped with their corresponding Borel σ-algebras. The above construction to define a σ-algebra in $\mathbb{R}^n$ from the rectangles sets in $\mathbb{R}$ can be extended to abstract measurable spaces and its formalisation is due to the monotone class theorem [150, Chapter 3]. Indeed, let $(\Delta_i, \mathcal{F}_i)$, $i = 1, \dots, n$, be measurable spaces then we can define the product measurable space $(\prod_{i=1}^n \Delta_i, \sigma(\prod_{i=1}^n \mathcal{F}_i))$ by means of a similar procedure.

[3]This topology can be constructed by means of the open intervals (a, b), with $a < b$, by taking all possible finite intersections of such sets and then taking an arbitrary union. This process is useful to construct topologies from simpler sets (see [99], for more details).

Definition 12. *Let $(\Delta, \mathcal{F})$ and $(\Omega, \mathcal{G})$ be measurable spaces. A function $f : \Delta \to \Omega$ is said to be measurable if, for all $G \in \mathcal{G}$, we have that $f^{-1}(G) \in \mathcal{F}$, that is, f^{-1} maps elements in $\mathcal{G}$ to elements in $\mathcal{F}$.*

Note that the definition of measurable functions resembles that of continuous functions given in Definition 4; in fact, these concepts are related. Let $f : \mathbb{R}^n \to \mathbb{R}^m$, with $\mathbb{R}^n$ and $\mathbb{R}^m$ both equipped with their Borel σ-algebras, then every continuous function is measurable but the converse does not hold, as measurable functions can be discontinuous (e.g., step functions are measurable but not continuous. We refer the reader to [130] for more details).

2.2.1 Probability measure spaces

At the core of probability theory is the concept of a probability measure, which is a set function that assigns a number in the interval $[0, 1]$ to subsets of the space Δ. Such a function must be countably additive, that is, the value it assigns to two disjoint sets must be the sum of the values it assigns to each individual set, and this property must hold also for any countable union of disjoint sets. It is impossible[4] to define a probability measure with these properties for all subsets of the real line; however, this can be done for σ-algebras and the resulting mathematical structure leads to the concept of a probability measure space.

Definition 13. *Let $(\Delta, \mathcal{F})$ be a measurable space. A probability measure on $(\Delta, \mathcal{F})$ is a set function $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying*

i) $\mathbb{P}\{\Delta\} = 1$.

ii) *$\mathbb{P}$ is σ-additive, i.e., for all sequences of pairwise disjoint subsets $F_i \in \mathcal{F}$, $i \in \mathbb{N}$, we have that*

$$\mathbb{P}\left\{\cup_{i\in\mathbb{N}} F_i\right\} = \sum_{i\in\mathbb{N}} \mathbb{P}\left\{F_i\right\}.$$

A probability measure space is the triple $(\Delta, \mathcal{F}, \mathbb{P})$ consisting of a set Δ, a σ-algebra $\mathcal{F}$, and a probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$.

[4] A full discussion of this impossibility involves the axiom of choice. See [127, Chapters 1 and 2] for more details.

The probability measure space $(\Delta, \mathcal{F}, \mathbb{P})$ provides the mathematical structure upon which a theory of probability can be developed. Indeed, a powerful integration theory, called Lebesgue integration, can be developed[5]. Standard concepts in probability can be adapted to this measure theoretic framework. A random variable is a measurable function from the probability measure space $(\Delta, \mathcal{F}, \mathbb{P})$ to the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, for continuous random variables, and to the measurable space $(\mathbb{Z}, 2^{\mathbb{Z}})$, for discrete random variables,where the notation 2^X represents the power set of X. Suppose X is a continuous random variable[6], then its distribution function is defined as

$$F_X(x) = \mathbb{P}\{X \leq x\} = \mathbb{P} \circ X^{-1}(-\infty, x], \tag{2.2}$$

also known as the *pushforward measure* of $\mathbb{P}$ through the map X. It is a fact that $F_X(x)$ is right-continuous and defines a probability measure on $\mathbb{R}$ as $m_{F_X}((a,b]) = F_X^-(b) - F_X(a)$, where $F_X^-(b) = \lim_{x \uparrow b} F_X(x)$ is the limit from the left of the function F_X. Note that $m_{F_X}(\{a\}) = 0$ whenever the distribution F_X is continuous at a and that continuity of F_X is guaranteed at points $a \in \mathbb{R}$ where $\mathbb{P} \circ X^{-1}(a) = 0$.

The expectation of a non-negative random variable is given by

$$\mathbb{E}\{X\} = \int_\Delta X(\delta) d\mathbb{P}(\delta) = \int_0^\infty \mathbb{P}[X > t] dt = \int_0^\infty x dF_X(x). \tag{2.3}$$

Throughout this thesis, we also invoke the following proposition.

Proposition 3 ([150], Chapter 6). *Let $(\Delta, \mathcal{F}, \mathbb{P})$ be a probability measure space, $X : \Delta \to [0, \infty)$ be a non-negative random variable with distribution given by $F_X : [0, \infty) \to [0, 1]$, and $\phi : [0, \infty) \to \mathbb{R}$ be a differentiable function. Then, we have that*

$$\mathbb{E}\{\phi(X)\} = \int_\Delta \phi(X(\delta)) d\mathbb{P}(\delta) = \int_0^\infty \phi(s) dF_X(s)$$

Proposition 3 connects the abstract definition of integration in the space Δ to an integral involving the distribution of the random variable X.

[5]For the sake of brevity, we will not delve into the details of the definition of Lebesgue integration theory. A rigorous construction of such a theory can be found in [127], [130], [150].

[6]Note the abuse of notation. In Section 2.1 we have used X to denote a non-empty set for which we have defined a topology. We hope this is clear from the context.

2.3 Optimisation

Optimisation is crucial to modern engineering applications. It involves minimising (or maximising) a function defined in a vector space – in this thesis, we only consider functions defined in the Euclidean space – over a subset of its domain. A template for an optimisation problem is given by [7]

$$\begin{aligned} \underset{x \in \mathcal{X}}{\text{minimise}} \quad & f(x) \\ \text{subject to} \quad & h(x) \leq 0, \end{aligned} \tag{2.4}$$

where[8] $f, h : \mathbb{R}^n \to \mathbb{R}$ are real-valued functions and $\mathcal{X}$ is a subset of $\mathbb{R}^n$ contained in the domain of f and h. Points x in the domain of f such that $h(x) \leq 0$ are called feasible points. The optimal set of (2.4) is composed of points $x^\star$ such that $f(x^\star) \leq f(x)$ for all x in the feasible set. Note that, by definition, the function f is constant on the optimal set and we denote by $f^\star$ this optimal value.

There are a plethora of iterative algorithms that, starting with a feasible point, produce a sequence converging to the optimal set associated to (2.4), and these include gradient-based, dual methods, primal-dual methods, among others [9], [11], [13], [21]. If functions f and h that compose problem (2.4) are convex (as defined below) and the set $\mathcal{X}$ is convex (please refer to Definition 9) we say that (2.4) is a convex optimisation problem; otherwise, we call (2.4) a non-convex optimisation problem.

2.3.1 Convex functions, subgradients

Throughout this thesis we will be dealing with convex optimisation problems. To define such problems, let us introduce the concept of convex functions.

[7]Throughout this thesis we assume implicitly that there exists a minimiser for all optimisation problems we are dealing with. This can be guaranteed, for instance, if the function f in (2.4) has sublevel sets that are closed (i.e., the set $\{x \in \mathbb{R}^n : f(x) \leq a\}$ is closed for all $a \in \mathbb{R}$) and if $\lim_{\|x\| \to \infty} f(x) = \infty$. For more details, please refer to [4], Chapters 2 and 3.

[8]We can also consider optimisation problems in which several inequality constraints are present, in this case we have $h : \mathbb{R}^n \to \mathbb{R}^p$, where p represents the number of inequality constraints. This more general case is omitted for brevity. The reader is referred to [21] for more details.

Definition 14. *A function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is said to be convex if the set*

$$\text{epi}(f) = \{(x, t) \in \mathbb{R}^n \times [-\infty, +\infty] : f(x) \leq t\}$$

is a convex set of $\mathbb{R}^{n+1}$.

The set epi(f) defined above is called the epigraph of f, and Definition 14 shows explicitly the connection between convex functions and convex sets, i.e., a function is convex if and only if its epigraph is convex. Note that in Definition 14 the function f can take values ∞ and $-\infty$ and, in principle, the undefined operation $\infty - \infty$ may happen. To this end, it is common to restrict attention to the effective domain of f, defined as

$$\text{dom}(f) = \{x \in \mathbb{R}^n : f(x) < \infty\}.$$

Hence, as long as $x \in \text{dom}(f)$ we cannot have $\infty - \infty$. Definition 14 entails the fact that f is convex if and only if

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y), \quad \text{for all } \theta \in (0, 1), \tag{2.5}$$

whenever $x, y \in \text{dom}(f)$, or $f(x) > \infty$ for all $x \in \mathbb{R}^n$. Let $C \subset X$ be a convex set, then its indicator function is defined as

$$1_C(x) = \begin{cases} 0, & \text{if } x \in C \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.6}$$

Note that, under Definition 14, the indicator functions of convex sets are convex.

Convex functions are continuous in the interior of their effective domain. The key idea to prove this is to show that if $\bar{x} \in \text{int}(\text{dom}(f))$, then we can use inequality (2.5) to show that f is locally Lipschitz, i.e., $|f(x) - f(y)| \leq L\|x - y\|$ for all $x, y \in \mathbb{B}(\bar{x}, \epsilon)$ for some $\epsilon > 0$. However, a convex function may not be differentiable in the interior of its effective domain, e.g., the function $f(x) = |x|$ is convex but non-smooth at the origin and this can bring some difficulties when relying on algorithms that require the computation of derivatives. Fortunately, convex functions possess at each point in the interior of its effective domain a generalised gradient vector, called the subgradient of f.

Proposition 4 ([10], Proposition 5.4.1)**.** *Let $f : \mathbb{R}^n \to (-\infty, \infty)$ be a convex function and $\bar{x} \in \operatorname{int}(\operatorname{dom}(f))$. Then, there exists a vector $g \in \mathbb{R}^n$ such that the inequality*

$$f(x) \geq f(\bar{x}) + g^\top (x - \bar{x}), \text{ for all } x \in \operatorname{int}(\operatorname{dom}(f)).$$

holds. The set of all such a g is called the subdifferential of f at $\bar{x}$ and is denoted by $\partial f(\bar{x})$.

Proof. We first prove the result for $n = 1$. To this end, let $\bar{x} \in \operatorname{int}(\operatorname{dom}(f))$ be fixed and consider $a < \bar{x} < b$ with $a, b \in \operatorname{int}(\operatorname{dom}(f))$. Under these choices and by convexity of f, we obtain that

$$\bar{x} = \theta a + (1 - \theta) b, \text{ for some } \theta \in (0, 1), \text{ and } f(\bar{x}) \leq \theta f(a) + (1 - \theta) f(b). \tag{2.7}$$

Manipulating (2.7), we obtain

$$\frac{f(\bar{x}) - f(a)}{\bar{x} - a} \leq \frac{f(b) - f(\bar{x})}{b - \bar{x}} \leq \frac{f(b) - f(a)}{b - a}, \text{ for all } a < \bar{x} < b. \tag{2.8}$$

Let $b > \bar{x}$ be fixed and use the left-most inequality in (2.8) to obtain

$$f(b) \geq f(\bar{x}) + (b - \bar{x}) \frac{f(\bar{x}) - f(a)}{\bar{x} - a}, \text{ for all } a < \bar{x}.$$

The result will be proved for $b > \bar{x}$ if we show that the limit from the left at $\bar{x}$ of $\frac{f(\bar{x}) - f(a)}{\bar{x} - a}$, i.e., $\lim_{a \uparrow \bar{x}} \frac{f(\bar{x}) - f(a)}{\bar{x} - a}$, converges. Note that since $\bar{x}$ and b are fixed, we know that this quantity is upper-bounded by the left-most inequality in (2.8). To show that the above limit exists it suffices proving the following monotonicity property

$$\frac{f(\bar{x}) - f(a)}{\bar{x} - a} \leq \frac{f(\bar{x}) - f(a')}{x - a'}, \text{ for all } a < a' < \bar{x}. \tag{2.9}$$

This can be done by (2.8), replacing $\bar{x}$ by a' and b by $\bar{x}$. Then we conclude that the limit exists. Define $g^- = \lim_{a \uparrow \bar{x}} \frac{f(\bar{x}) - f(a)}{\bar{x} - a}$.

We now try to show the result for $a < \bar{x}$. To this end, we fix $a < \bar{x}$ and consider the left-most inequality in (2.8) to obtain

$$f(a) \geq f(\bar{x}) + (a - \bar{x}) \frac{f(b) - f(\bar{x})}{b - \bar{x}}, \text{ for all } b > \bar{x}.$$

Similarly as before, the left-most inequality in (2.8) – in which $\bar{x}$ and a are fixed – implies that the limit $\lim_{b\downarrow\bar{x}} \frac{f(b)-f(\bar{x})}{b-\bar{x}}$ is bounded from below. And to show that such a limits exists it suffices to prove

$$\frac{f(b')-f(\bar{x})}{b'-\bar{x}} \leq \frac{f(b)-f(\bar{x})}{b-\bar{x}}, \text{ for all } b' < b.$$

Proceeding as before, this can be done with by replacing a to $\bar{x}$ and $\bar{x}$ to b' in (2.8). Let $g^+ = \lim_{b\downarrow\bar{x}} \frac{f(b)-f(\bar{x})}{b-\bar{x}}$. To prove the result it remains to be shown that $g^+ = g^-$. To this end, we use the both inequalities in (2.8). First, take the limit with respect to b and note that this yields

$$\frac{f(\bar{x})-f(a)}{\bar{x}-a} \leq g^+ \leq \frac{f(\bar{x})-f(a)}{\bar{x}-a},$$

now taking the limit with respect to a implies that $g^- \leq g^+ \leq g^-$, thus yielding $g^+ = g^- = g$. This concludes the proof of the proposition for $n = 1$. The proof for general n can be obtained from the case $n = 1$ and is omitted for brevity. □

Proposition 4 states that the subdifferential is non-empty in the interior of the effective domain of a convex function. For instance, let $f : \mathbb{R} \to [0, \infty)$ be defined as

$$f(x) = \begin{cases} |x|, & \text{if } x \in [-1, 1] \\ +\infty, & \text{otherwise.} \end{cases}$$

The interior of the effective domain is $(-1, 1)$ and the subdifferential at each point is given by

$$\partial f(x) = \begin{cases} -1, & \text{if } -1 < x < 1 \\ [-1, 1], & \text{if } x = 0 \\ 1, & \text{if } 0 < x < 1, \end{cases}$$

which is non-empty, thus in accordance with Proposition 4. Note that Proposition 5.4.1 in [10] also implies that the resulting subdifferential is compact at each point in the interior of the domain of f.

2.3.2 Duality theory

As stated at the beginning of this section, given an optimisation problem, our main goal is to produce an iterative scheme that converges to the optimal set of (2.4). In this process, sometimes it is convenient to study a related optimisation problem, called the dual optimisation problem, and recover the optimal solution of (2.4) by means of the optimal solution to the dual problem. To this end, we consider the Lagrangian function associated to (2.4)

$$L(x, \lambda) = f(x) + \lambda h(x), \tag{2.10}$$

where λ is a non-negative scalar. For every $\lambda \geq 0$ and feasible point x of (2.4), we have that

$$L(x, \lambda) \leq f(x)$$

which then implies $\inf_{x \in \mathcal{X}} L(x, \lambda) \leq f^\star$, where $f^\star$ is the optimal value of (2.4). Let $v(\lambda) = \inf_{x \in \mathcal{X}} L(x, \lambda)$. The previous argument then shows that $v(\lambda) \leq f^\star$ for all $\lambda \geq 0$. In other words, the optimal value of the optimisation problem

$$\underset{\lambda \geq 0}{\text{maximise}} \quad v(\lambda), \tag{2.11}$$

which is denoted by $v^\star$, is a lower bound to $f^\star$, i.e., we have that $v^\star \leq f^\star$. Inequality $v^\star \leq f^\star$ is referred to as weak duality and holds irrespective of (2.4) being convex or not (for more details, please refer to [21, Chapter 5] or [10, Chapter 4]). Whenever $v^\star = f^\star$ we say that strong duality holds. A sufficient condition for this to happen is if f and h are convex and there exist a feasible point in the interior of the domain of f such that $h(x) < 0$. The latter condition is called Slater's condition.

2.3.3 Chance-constrained optimisation

In this section, we introduce a formulation of optimisation problems with uncertain constraints. Let $(\Delta, \mathcal{F}, \mathbb{P})$ be a probability space (see Section 2.2). Fix an upper bound for the probability of constraint violation $\epsilon \in (0,1)$, then the chance-constrained formulation of an optimisation problem is given by

$$\begin{aligned} \underset{x \in \mathcal{X}}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & \mathbb{P}\{\delta : g(x,\delta) > 0\} \leq \epsilon, \end{aligned} \tag{2.12}$$

where $\mathcal{X} \subset \mathbb{R}^d$ is a closed, convex set with non-empty interior, $c \in \mathbb{R}^d$ is a vector, and function[9] $g : \mathbb{R}^d \times \Delta \to \mathbb{R}$ is measurable[10] in the second argument for each $x \in \mathcal{X}$. The feasible set of (2.12) is composed of points $x \in \mathbb{R}^d$ such that the probability that a sample δ drawn from $\mathbb{P}$ violates the constraint $g(x,\delta) \leq 0$ is smaller than ϵ.

It is known that the feasible set of (2.12) may be non-convex, even when the function $g(\cdot,\delta) : \mathbb{R}^d \to \mathbb{R}$ is convex for all $\delta \in \Delta$ [117], [118], so (2.12) is in general hard to solve. Chapters 5 and 6 of this thesis address an approximation of (2.12) that is based on independent and identically distributed samples from the uncertain parameter $\delta \in \Delta$.

2.4 Compression learning

In this section, we briefly introduce some concepts related to statistical learning theory and prove for completeness the main result of [94], which constitutes the basis of the results presented in Chapters 5 and 6.

The learning problem we are interested in can be formulated in terms of the probability measure space $(\Delta, \mathcal{F}, \mathbb{P})$. We aim to learn a subset of the uncertainty space Δ, called the target set and denoted by T, by means of m-independent and identically distributed (i.i.d.) samples $S = \{\delta_1, \ldots, \delta_m\}$. Due to the samples being i.i.d, we can consider S as an element in the probability measure space $(\Delta^m, \sigma(\prod_{i=1}^m \mathcal{F}), \mathbb{P}^m)$ (refer to Section 2.2 for a brief introduction to this construction). Throughout this thesis we may consider S as a subset of Δ, writing $S \subset \Delta$, or as an element of Δ^m, writing $S \in \Delta^m$.

[9]Note the overlapping notation. In Section 2.3.1 g denoted the subgradient of a convex function, whereas here g denotes a measurable function. We hope this is clear from the context.

[10]This implies that the set $\{\delta \in \Delta : g(x,\delta) > 0\} = g^{-1}(x, A)$, with the Borel set $A = (0, \infty)$, is an element of $\mathcal{F}$, hence rendering (2.12) well-defined, as $\mathbb{P}$ is defined only on measurable sets. See Sections 2.1 and 2.2 as well as [130] for more details about this construction.

We assume that an oracle that labels each element of the unknown target set T is available. Our goal is to approximate T by using a mapping $\mathcal{A} : \Delta^m \to 2^\Delta$ that takes as input the available samples S and outputs a subset of Δ, represented by $\mathcal{A}(S)$, and referred to as the hypothesis set. In other words, based on the available information encoded in S, we aim to create an approximation of the target space T through the subset $\mathcal{A}(S)$. It is worth mentioning that we are defining the mapping $\mathcal{A}$ for any cardinality m, even though this is not explicity mentioned in our notation. Under this convention, observe that we can make sense of $\mathcal{A}(C)$ for any $C \subset S$. We will come back to this point later.

To measure the distance between the target and the hypothesis set we use

$$d_{\mathbb{P}}(T, \mathcal{A}(S)) = \mathbb{P}\{(T \setminus \mathcal{A}(S)) \cup (\mathcal{A}(S) \setminus T)\},$$

which measures the probability of the symmetric distance between T and $\mathcal{A}(S)$ (see [147, Chapter 1] for more details). The mapping $\mathcal{A}$ is said to be probably approximately correct (PAC) if for all $\epsilon \in (0, 1)$ and for all $m \in \mathbb{N}$ there exists a function $q : \mathbb{N} \times [0, 1] \to \mathbb{R}$ such that

$$\mathbb{P}^m\{S \in \Delta^m : d_{\mathbb{P}}(\mathcal{A}(S), T) > \epsilon\} \le q(m, \epsilon), \text{ with } \lim_{m\to\infty} q(m, \epsilon) = 0. \tag{2.13}$$

Inequality (2.13) is required to hold irrespective of the probability measure $\mathbb{P}$, and hence typical results in PAC learning are distribution-free.

The following definition is key to establish sufficient conditions that guarantee the existence of bounds as in (2.13).

Definition 15 (Compression set)**.** *Fix $m \in \mathbb{N}$, and consider $S \in \Delta^m$. Let $\zeta < m$, and $C \subset S$ with cardinality $|C| = \zeta$. Consider a mapping $\mathcal{A} : \Delta^m \to 2^\Delta$. If*

$$\delta \in \mathcal{A}(C), \quad \textit{for all } \delta \in S,$$

then C is called a compression set of cardinality ζ for $\mathcal{A}$.

In other words, a compression set C contains sufficient information to generate a subset $\mathcal{A}(C)$ of Δ that contains all the samples in S. This latter property is called consistency within the learning theoretic literature.

Remark 1. *As stated earlier, the mapping $\mathcal{A}$ is usually defined for any cardinality of the set S. For instance, let*

$$\mathcal{A}(S) = \{\delta \in \Delta : g(x^\star(S), \delta) > 0\}, \tag{2.14}$$

where $x^\star(S)$ is the optimal solution of

$$\begin{aligned} \underset{x}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta) \le 0, \ \textit{for all } \delta \in S. \end{aligned} \tag{2.15}$$

The optimisation problem (2.15) has one constraint for each element of S and is called a scenario program (please refer to Chapters 5 and 6 for more details). For any $C \subset S$, the mapping $\mathcal{A}(C)$ is given by (2.14) with $x^\star(S)$ being replaced by $x^\star(C)$, the solution of (2.15) with only the constraints generated by the samples in C being enforced. We do not index the mapping $\mathcal{A}$ by the cardinality of its input set as this will be marginal for our results and would only introduce unnecessary notation.

The notion of compression sets can be used to derive PAC bounds that quantify the confidence with which $\mathcal{A}(S)$ is an approximation for the target set T. Specifically, we are interested in establishing uniqueness and existence of a compression set associated with $\mathcal{A}$ to provide the following PAC result.

Theorem 1 ([94], Theorem 3)**.** *Let $T \subset \Delta$ be an unknown target set. Consider m independent samples, $S = \{\delta_1, \ldots, \delta_m\}$, from $\mathbb{P}$ and suppose that there exists a unique compression set of size $\zeta < m$ associated to a mapping $\mathcal{A} : \Delta^m \to 2^\Delta$. Then for any $\epsilon \in (0, 1)$ we have that*

$$\mathbb{P}^m\{S \in \Delta^m : d_{\mathbb{P}}(\mathcal{A}(S), T) > \epsilon\} = \sum_{i=0}^{\zeta-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}. \tag{2.16}$$

Proof. Let $E = \{S \in \Delta^m : d_{\mathbb{P}}(\mathcal{A}(S), T) > \epsilon\} \subset \Delta^m$ and $V : \Delta^m \to [0, 1]$ be the random variable

$$V(S) = d_{\mathbb{P}}(T, \mathcal{A}(S)). \tag{2.17}$$

Our goal is to compute $\mathbb{P}^m\{E\}$, which is the tail of the distribution of the random variable $V : \Delta^m \to [0, 1]$ in (2.17). To this end, we use existence and uniqueness of a compression set to produce a partition for the uncertainty space Δ^m. In fact, note that if there exists a unique compression set, then the collection of subsets given by

$$A_J = \{S \in \Delta^m : \mathcal{A}(J) \text{ contains all the samples in } S\},\ J \subset S,\ |J| = \zeta,$$

partitions the space Δ^m since A_I and A_J are disjoint sets for $I \neq J$ with $I, J \subset S$ having cardinality ζ (due to uniqueness), and $\cup_{\substack{J\subset S\\|J|=\zeta}} A_J$ is equal to Δ^m (due to existence). Since $(\Delta^m, \sigma(\prod_{i=1}^m \mathcal{F}), \mathbb{P}^m)$ is a probability measure space, we have that

$$\sum_{\substack{J\subset S\\|J|=\zeta}} \mathbb{P}^m\{A_J\} = \mathbb{P}^m\left\{\cup_{\substack{J\subset S\\|J|=\zeta}} A_J\right\} = \mathbb{P}^m\{\Delta^m\} = 1, \tag{2.18}$$

where the first equality is due to the fact that the collection $(A_J)_{\substack{J\subset S\\|J|=\zeta}}$ is pairwise disjoint, the second to the fact that the union of such a collection is the whole space, and the third due to $(\Delta^m, \sigma(\prod_{i=1}^m \mathcal{F}), \mathbb{P}^m)$ being a probability measure space. Besides, as the mapping $\mathcal{A}(S)$ is permutation invariant, we conclude that the events A_J are equally likely. By (2.18), we then obtain that[11]

$$\binom{m}{\zeta}\mathbb{P}^m\{A_J\} = \binom{m}{\zeta}\mathbb{E}\{A_J\} = \binom{m}{\zeta}\mathbb{E}\{\mathbb{E}\{A_J|V(S)\}\}, \tag{2.19}$$

where the last equality follows from the tower property (or Radon-Nikodym's theorem, see Chapters 9 and 14 in [150]) by conditioning under the random variable $V(S)$ in (2.17). Using the consistency property in the definition of the compression set we have that

$$\mathbb{E}\{A_J|V(S)\} = (1 - V(S))^{m-\zeta},$$

due to the i.i.d. assumption on the samples in S and the fact that all samples not in A_J must be in $\mathcal{A}(S) \cap T$. Hence, from (2.18) and (2.19), we obtain

$$\begin{aligned} 1 &= \binom{m}{\zeta}\mathbb{E}\{\mathbb{E}\{A_J|V(S)\}\} = \binom{m}{\zeta}\int_{\Delta^m}(1 - V(\delta))^{m-\zeta}d\mathbb{P}^m(\delta) \\ &= \binom{m}{\zeta}\int_{\Delta^m}\phi(V(\delta))d\mathbb{P}^m(\delta) = \binom{m}{\zeta}\int_{\Delta^m}(1 - s)^{m-\zeta}dF(s), \end{aligned} \tag{2.20}$$

[11] The notation $\mathbb{E}\{A_J\}$ for a set A_J indicates the expectation of the indicator function 1_{A_j} of A_J.

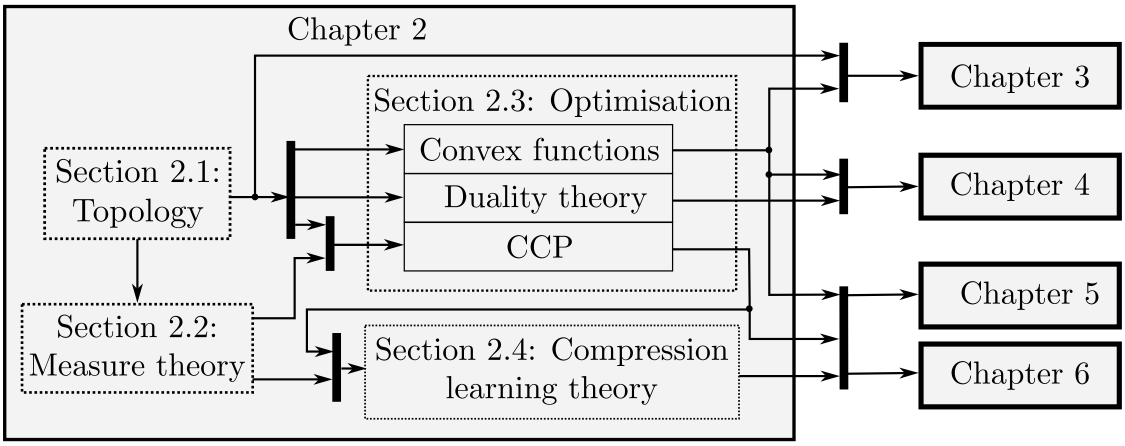


Figure 2.1: Schematic diagram with the interdependence of the results of this chapter with the subsequent ones in this thesis.

where $\phi(s) = (1-s)^{m-\zeta}$ and $F(s) = \mathbb{P}^m \circ V^{-1}(0, s]$ is the distribution function of the random variable $V(S)$. To obtain the last equality in (2.20) we have used Proposition 3. Hence, the distribution of V satisfies the following moment problem

$$\binom{m}{\zeta} \int_{\Delta^m} (1-s)^{m-\zeta} dF(s) = 1, \text{ for all } m \geq \zeta. \tag{2.21}$$

As well noticed in [29], the unique solution to (2.21) is the distribution given by $F(s) = s^\zeta$. To conclude the proof of this theorem, we use the total law of probability to obtain

$$\begin{aligned}\mathbb{P}^m\{E\} &= \sum_{\substack{J\subset S\\ |J|=\zeta}} \mathbb{P}^m\{A_J \cap E\} = \binom{m}{\zeta}\mathbb{E}\{A_J \cap E\} = \binom{m}{\zeta}\mathbb{E}\{\mathbb{E}\{A_J | V(S), E\}\} \\ &= \binom{m}{\zeta} \int_\epsilon^1 (1-s)^{m-\zeta} dF(s) = \sum_{i=0}^{\zeta-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i},\end{aligned} \tag{2.22}$$

where in the fourth equality we have again used Proposition 3, and in the fifth inequality we have used the fact that $F(s) = s^\zeta$. The details of the latter computation have been omitted for brevity but can be found in [29]. This concludes the proof of the theorem. □

2.5 Further reading

For the sake of brevity we have only presented in this chapter the minimal content to make this thesis self-contained if we assume a full knowledge on linear algebra, control theory, and real analysis. The topological concepts of Section 2.1, though sufficient for our purposes, are condensed and a full treatment of the topic can be found in [99] for an introduction and in [113] for a more advanced treatment. As for Section 2.2, we refer to excellent books [127], [130], [150] for a thorough treatment of the subject. Additional material to support Section 2.3 includes [10], [21], for a introduction to the subject, and [4], [122], [138], for a more advanced presentation. The result of Section 2.4 can be found in [94]. For related results on statistical learning theory, the reader is referred to [51], [146] and to the book [147] and references therein.

2.6 Mathematical tools in this thesis

Sections 2.1 and 2.2, though not strictly essential for an understanding of the results presented in this thesis, provide us with a solid mathematical foundation towards the results here presented.

To clarify how the concepts presented in this chapter are related to subsequent chapters, we present the diagram in Figure 2.1. Topological concepts are important to understand some technicalities related to optimisation problems and developments of technical lemmas in Chapter 3. These also lead to a proper understanding on some measure theoretic concepts that then form the basis for chance-constrained problems (CCP) and, consequently, all the results in Sections 2.4 and Section 2.2, and Chapters 5 and 6.

Observe in the diagram of Figure 2.1 that all subsections of the Section 2.3 have been highlighted so that their role in subsequent chapters is emphasised. For instance, we can notice that Subsection 2.3.1 on convex functions is crucial to all subsequent chapters, while Subsection 2.3.3 on chance-constrained optimisation is related to the results in Chapters 5 and 6 and Subsection 2.3.2 on duality theory is related to Chapter 4.

3 Subgradient averaging for multi-agent optimisation

This chapter addresses the problem of scalability in optimisation algorithms mentioned in Chapter 1 of this thesis. We rely on distributed optimisation over multi-agent networks that leverages distributed computation to solve large-scale problems.

3.1 Introduction

Distributed optimisation deals with multiple agents interacting over a network and has found numerous applications in different domains, such as wireless sensor networks [5], [98], robotics [95], and power systems [19], due to its ability to parallelize computation and prevent agents from sharing information considered as private. Typically, distributed algorithms are based on an iterative process in which agents maintain some estimate about the decision vector in an optimisation context, exchange this information with neighbouring agents according to an underlying communication protocol/network, and update their estimate on the basis of the received information.

Despite the intense research activity in this area, only a few algorithms can simultaneously deal with time-varying networks, non-differentiable objective functions and account for the presence of constraints [82], [92], [100], [152], [155], features that are often treated separately in the literature.

In this chapter, we study a class of optimisation problems that involves a separable objective function, while the feasible set can be decomposed as an intersection of different compact convex sets. A similar algorithm to the one presented in this chapter can be found in [67]; however, no analysis for the particular setting we are considering is presented. Besides, references [78], [84] characterize the convergence rate of a sub-gradient algorithm under different constraint sets per agent that does not possess subgradient averaging, and [92], [155] show asymptotic convergence of distributed algorithms with different constraint sets and time-varying communication network. Hence, by combining (sub)-gradient averaging and providing an analysis that yields convergence rates under time-varying communication networks and different constraint sets per agent, the results in this chapter are distinct from all the above literature.

Another closely related algorithm to the one presented here is the one in [91]. This provides convergence rates assuming a regularity condition on the local sets (weaker than compactness) and requiring the network to be row-stochastic; however, it does not analyse the case where the communication network is time-varying. This requires different analysis arguments, thus complementing the results in [91], extending them to allow for time-varying networks.

Although only marginally related to the results of this chapter, it is worth mentioning distributed algorithms that deal with similar optimization problems [120], [133], [137]. Paper [133] proposes an algorithm whose convergence is valid for non-convex objectives and directed communication network, while [120], [137] use a constant step size to establish linear convergence rates for strongly convex functions. Moreover, distributed algorithms based on proximal methods with constant step sizes have been proposed in [38]. In this setting, the objective function is assumed to be differentiable to obtain convergence to the optimal solution of problem (3.1), and the size of the allowable step-size is upper bounded by a quantity related to

Table 3.1: Summary of distributed schemes for smooth and non-smooth optimisation.

	Smooth + Constant step-size				Non-smooth + Diminishing step-size			
	Common sets		Different sets		Common sets		Different sets	
	Convex	Strongly Convex	Convex	Strongly Convex	Convex	Strongly Convex	Convex	Strongly Convex
No (sub)grad. avg.	[64], [101], [153]	[153]	[79]	-	[100], [133]	[85], [142]	[78], [84], [92], [155]	-
(Sub)grad. avg.	[120], [133], [137]	[120], [133], [137]	-	-	[46], [82], [133], [152]	-	our work, [91]	-

the Lipschitz constant of the objective function. Unlike these results, we allow for non-differentiable objective functions.

To better position this chapter within the recent literature, we summarise the main distributed algorithms that are amenable to smooth and non-smooth constrained optimisation in Table 3.1. We highlight both scenarios of common and different local constraint sets, which are indicated in the table by common sets and different sets, respectively. In this brief summary, we restrict our attention to algorithms that use constant step size for smooth optimisation, and to those that use diminishing step sizes for the non-smooth case. We also present a categorisation of these schemes between those that have results for general convex functions and strongly convex functions. In row entitled “No (sub)grad. avg.”, we include distributed algorithms based on projected (sub)gradient, proximal minimisation, and primal-dual update that do not leverage on averaging first-order information from neighbouring agents. In contrast, row “(Sub)grad. avg.” includes algorithms that exploit (sub) gradient averaging. Among the few algorithms that are suitable for different local sets, the one proposed in this chapter is the first to possess a convergence rate that matches that of the common local sets case, and simultaneously allows agents to use first-order information of their neighbours under time-varying communication networks, and may speed up practical convergence. The main contribution of this chapter is the introduction and the characterisation of the convergence rate for a new subgradient averaging algorithm. The proposed scheme allows us to account for time-varying networks, non-differentiable objective functions and different constraint sets per agent as in [92], while achieving faster practical convergence as it is based on subgradient averaging as in [46], [67], [91]. We highlight that allowing simultaneously for different constraint sets per agent and time-varying communication network by means of a subgradient averaging scheme is a distinct feature of the algorithm proposed in this chapter.

This chapter is organised as follows. In Section 3.2 we present the problem statement and main assumptions, followed by a numerical construction that motivates the algorithm of this chapter. In Section 3.3 we present the proposed scheme and the main convergence results, namely, asymptotic convergence in iterates and a convergence rate as far as the optimal value is concerned. In Section 3.4 we study the robust linear regression problem and ℓ_2 regression with regularisation to demonstrate the main algorithmic features of our scheme and to compare our strategy against existing methods. Finally, the connection between the results in this chapter with the main problems stated in Chapter 1 is presented in Section 3.5. All the proofs have been deferred to Section 3.6.

3.2 Problem statement and a motivating example

3.2.1 Problem set-up and network communication

Consider the optimisation problem

$$\begin{aligned} \underset{x}{\text{minimise}} \quad & f(x) = \sum_{i=1}^{m} f_i(x) \\ \text{subject to} \quad & x \in \cap_{i=1}^{m} X_i, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, and $f_i : \mathbb{R}^n \to \mathbb{R}$ and $X_i \subset \mathbb{R}^n$ constitute the local objective function and constraint set, respectively, for agent i, $i = 1, \dots, m$. We suppose that each agent i possesses as private information the pair (f_i, X_i) and maintains a local estimate x_i of the common decision vector x.

The goal is for all agents to agree on the local variables, that is, $x_i = x^\star$, for all $i = 1, \dots, m$, where $x^\star$ is an optimiser of (3.1), i.e., a feasible point such that $f(x^\star) \leq f(x)$ for all $x \in \cap_{i=1}^{m} X_i$. Throughout this chapter we impose the following assumption:

Assumption 1. *We assume that:*

i) For all $i = 1, \dots, m$, the function f_i is convex.

ii) *The set $X_i \subset \mathbb{R}^n$ is compact and convex for all $i = 1, \ldots, m$, and $\cap_{i=1}^m X_i$ has a non-empty interior.*

iii) *The subgradient of the function $f(x)$ is bounded on $\cup_{i=1}^m X_i$, that is, $L = \max_{\substack{\xi \in \partial f(x), \\ x \in \cup_{i=1}^m X_j}} \|\xi\|_2 < \infty$, where $\partial f(x)$ represents the subdifferential of f at x.*

Assumption 1 imposes standard restrictions for constrained non-smooth optimisation. Item *ii)* implies informally that $\cup_{i=1}^m X_i$ has volume in $\mathbb{R}^n$, i.e., that the affine hull of $\cup_{i=1} X_i$ has dimension n. Moreover, the compactness assumption of item *ii)* guarantees that the optimal set of problem (3.1) is non-empty. Item *iii)* is an assumption that is needed to prove convergence of sub-gradient methods applied to problem (3.1). Under item *iii)*, the sub-gradient of the function f can be evaluated at points that belong to $\cup_{i=1}^m X_i$. We provide in Section 3.6.2 a technical condition on the domain of the functions f_i that is sufficient to guarantee that Assumption 1, item *iii)*, holds. An important consequence of Assumption 1 is given in the following lemma.

Lemma 1. *Under Assumption 1, we have that:*

i) *The set* $\text{conv}(\cup_{i=1}^m X_i)$ *is compact.*

ii) *The function f is Lipschitz continuous over $\cap_{i=1}^m X_i$, i.e., the following inequality hods*

$$|f(x) - f(y)| \leq L\|x - y\|_2, \quad \forall\ x, y\ \in\ \cap_{i=1}^m X_i,$$

where L is the constant defined in Assumption 1, item iii).

Typical choices of functions that satisfy Assumption 1 are piecewise-linear functions, quadratic convex functions and the logistic regression function.

In this chapter we concentrate in solving problem (3.1) through a network of agents that use only the available local information, namely, the pair (f_i, X_i) and the current estimate for the optimal solution, $x_i(k)$, $i = 1, \ldots, m$, maintained by agent i at a given instance k. We will show how $x_i(k), i = 1, \ldots, m$, can be constructed and updated in Section 3.3, with k playing the role of iteration index. Let us first characterise the underlying communication network. Let $\mathcal{G}(k) = (\mathcal{N}, \mathcal{E}(k))$ be an undirected graph, where $\mathcal{N} = \{1, \ldots, m\}$ is the number of agents and $\mathcal{E}(k) \subset \mathcal{N} \times \mathcal{N}$ is the set of edges at iteration k, that is, only if node $(j,i) \in \mathcal{E}(k)$ then node j sends information to node i at iteration k. We associate the time-varying matrix $A(k)$ to the edge set $\mathcal{E}(k)$, with $a_{ji}(k) > 0$ only if $(j,i) \in \mathcal{E}(k)$ at time k. As the graph is undirected, the matrix $A(k)$ can be chosen to be symmetric. We also define the graph $\mathcal{G}_\infty = (\mathcal{N}, \mathcal{E}_\infty)$, in which $(j,i) \in \mathcal{E}_\infty$ if agent j communicates with agent i infinitely often. We impose the following assumptions on the matrix $A(k)$.

Assumption 2. *We assume that:*

i) The graph $(\mathcal{N}, \mathcal{E}_\infty)$ is strongly connected. Moreover, there exits a uniform upper bound on the communication time for all $(j,i) \in \mathcal{E}_\infty$.

ii) There exists $\tau \in (0,1)$ such that for all $k \in \mathbb{N}$ and for all $i, j = 1, \ldots, m$, $a_{ii}(k) \geq \tau$, and if $a_{ji}(k) > 0$ then we have that $a_{ji}(k) \geq \tau$.

iii) Matrix $A(k)$ is doubly stochastic.

These are standard requirements in the distributed optimisation literature. We refer the reader to [46], [92], [102], [103] for more details.

3.2.2 Dealing with different constraint sets

Before proceeding to the presentation of the proposed algorithm, we would like to motivate the need to developing a different analysis to deal with different constraint sets. To this end, consider the iterative scheme[1]

$$z_i(k+1) = \sum_{j=1}^{m} a_{ji} z_j(k) + g_i(k) \tag{3.2a}$$

$$x_i(k+1) = \operatorname*{argmin}_{\xi \in X_i} z_i(k+1)^\top \xi + \frac{1}{c(k)} \|\xi\|_2^2, \tag{3.2b}$$

[1]It should be noted that $z_i, i = 1, \ldots, m$, in (3.2a) should not be confused with that of Step 2 in Algorithm 1 presented in the sequel; we use the same symbol to match the notation in [46] and ease the reader.

which consists of a modified version of the algorithm considered in [46], adapted to account for different constraint sets in each agent's local optimisation problem. In the setting of the previous section, notice that matrix A in (3.2a) corresponds to a fully-connected, time-invariant network. Assumption 2 is satisfied if the graph $(\mathcal{N}, \mathcal{E})$ is strongly connected and matrix A is doubly-stochastic.

Observe that (3.2a) constitutes a subgradient update step, with neighbouring local variables $z_j(k)$ being "mixed" according to the matrix A and added to $g_i(k) \in \partial f_i(x_i(k))$, i.e., a subgradient of f_i evaluated at $x_i(k)$, $i = 1, \dots, m$. Step (3.2b) is an optimisation program with the objective function being the sum (weighted via $c(k)$) of

$$z_i(k+1)^\top \xi\text{: linear “proxy” of } f_i,$$

and a regularization term $\|\xi\|_2^2$. To comply with [46], we set $c(k) = \frac{1}{\sqrt{k+1}}$. Recall that the algorithm in [46] involves the same constraint set in the update rule of (3.2b), that is $X_i = X$ for all $i = 1, \dots, m$, and possesses a guaranteed convergence rate of $\mathcal{O}(\frac{\ln k}{\sqrt{k}})$ for the running averages of the iterates $x_i(k)$; here, we introduce a different set X_i per agent and show that this (natural) modification may lead to erroneous results.

Consider a two-agent instance of (3.1), i.e., $m = 2$ with $x \in \mathbb{R}^2$, $f_i = x^\top Q x + q_i^\top x + r_i$, for $i = 1, 2$ and

$$Q = \begin{bmatrix} 1.2 & 0.4 \\ 0.4 & 1.8 \end{bmatrix}, \quad q_1 = \begin{bmatrix} 8 \\ -4 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 2.93 \\ -11.46 \end{bmatrix},$$
$$r_1 = 20, \quad r_2 = 25. \tag{3.3}$$

The local constraint sets are given by $X_1 = [-1, 1] \times [-1, 1]$ and $X_2 = [0.5, 2.5] \times [0.5, 2.5]$. The feasible set $X_1 \cap X_2$ is the box $[0.5, 1] \times [0.5, 1]$. Figure 3.1 depicts the level curves of the quadratic functions $f_1(x)$ (dashed-red lines), f_2 (double-dashed lines), and $f = f_1 + f_2$ (solid-black lines). The red and blue boxes represent the sets X_1 and X_2 respectively, with the feasible set, $X_1 \cap X_2$, being also indicated in the figure in black.

By inspection the optimal solution of f_1 under the constraint $x \in X_1$ is $\hat{x}_1^\star = [-1, 1]^\top$. Similarly, the optimal solution for f_2 under $x \in X_2$ is $\hat{x}_2^\star = [0.5, 2.5]^\top$. We then have the following proposition.

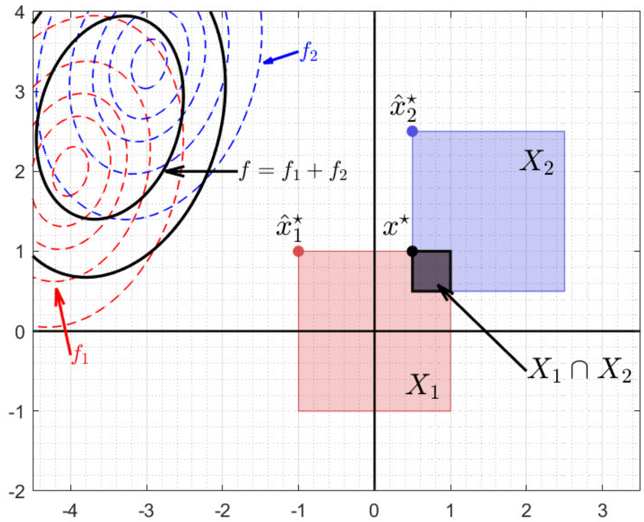


Figure 3.1: Geometric representation of problem instance encoded by (3.3). The red ellipsoids (dashed lines) correspond to the level curves of f_1, the blue ellipsoids (double-dashed lines) represent the function f_2, while the black (solid lines) ellipsoids to the ones of $f = f_1 + f_2$. The shaded red box illustrates the constraint set X_1, while the shaded blue box illustrates X_2. Vectors $\hat{x}_1^\star = [-1, 1]^\top$ and $\hat{x}_2^\star = [0.5, 2.5]^\top$ are the optimal solutions of $f_1(x)$ and $f_2(x)$ under the constraints X_1 and X_2, respectively. The global optimal solution of $f = f_1 + f_2$ with matrices given by (3.3) subject to $x \in X_1 \cap X_2$ is denoted by $x^\star$. This construction shows that $\hat{x}_1^\star$ and $\hat{x}_2^\star$ constitute fixed-points of (3.2) thus preventing the iteration from reaching $x^\star$ if initialised at those points.

Proposition 5. *Let* $(z_i(k))_{k\in\mathbb{N}}, (x_i(k))_{k\in\mathbb{N}}$, $i = 1, 2$, *be the sequences generated by algorithm* (3.2) *when applied to problem* (3.3) *with initial conditions* $x_i(0) = \hat{x}_i^\star$, $i = 1, 2$, *and with* $A = \frac{1}{2}\mathbf{1}\mathbf{1}^\top$ *and* $c(k) = \frac{1}{\sqrt{k+1}}$. *We have that*

$$x_1(k) = \hat{x}_1^\star, \quad x_2(k) = \hat{x}_2^\star, \quad \forall k \in \mathbb{N}.$$

Proposition 5 shows that $\hat{x}_1^\star$ and $\hat{x}_2^\star$ constitute fixed points of (3.2), hence the iteration cannot reach $x^\star$ if initialised from these points. This highlights the fact that not all algorithms that have been proposed so far can be extended to the setting where local constraint sets are different.

3.3 Distributed Methodology

3.3.1 Proposed algorithm

The main steps of the proposed scheme are summarized in Algorithm 1. We initialise each agents' local variable with an arbitrary $x_i(0) \in X_i$, $i = 1, \ldots, m$; such points are not required to belong to $\cap_{i=1}^m X_i$.

At iteration k, agent i receives x_j from the neighbouring agents and averages them through $A(k)$, which captures the communication network, to obtain $z_i(k)$. Recall that we denote the element of the j-th row and i-th column of matrix $A(k)$ by $a_{ji}(k)$. Agent i then calculates a subgradient, g_i, of its own objective function evaluated at $z_i(k)$ and broadcasts this information back to its neighbours. In the sequel, agent i averages the received $g_j(z_j(k))$ in order to compose a proxy for a subgradient of $f(x)$, namely, $d_i(k)$. Finally, agents minimise a linear proxy $d_i(k)^\top \xi$ of $f(\xi)$ plus a regularization term weighted by $\frac{1}{c(k)}$. An alternative interpretation of this last computation is that agents update their local estimates by performing a subgradient step with step size $c(k)$ and projecting $z_i(k) - c(k)d_i(k)$ onto their local set. Indeed, Step 4 in Algorithm 1 can be rewritten as

$$x_i(k+1) = \mathcal{P}_{X_i}[z_i(k) - c(k)d_i(k)]$$

where $\mathcal{P}_{X_i}[\cdot]$ denotes projection onto the set X_i.

Algorithm 1 Proposed distributed algorithm

Require: : $x_i(0), \quad i = 1, \ldots, m$

For $i = 1, \ldots, m$, **repeat until convergence**

1: Compute $z_i(k) = \sum_{j=1}^m a_{ji}(k)x_j(k)$,

2: Pick $g_i(z_i(k)) \in \partial f_i(z_i(k))$,

3: Compute $d_i(k) = \sum_{j=1}^m a_{ji}(k)g_j(z_j(k))$,

4: Compute $x_i(k+1) = \text{argmin}_{\xi \in X_i} d_i(k)^\top \xi + \frac{1}{2c(k)}\|z_i(k) - \xi\|_2^2$,

5: Set $k \leftarrow k+1$

end

3.3.2 Convergence results for Algorithm 1: square-summable step sizes.

Throughout this section, we impose the following assumption on the step size $c(k)$.

Assumption 3. *Let $(c(k))_{k\in\mathbb{N}}$ be the sequence adopted in Algorithm 1. We require that:*

i) $c(k)$ is non-negative and non-increasing;

ii) $\sum_{k=1}^{\infty} c(k) = \infty$ and $\sum_{k=1}^{\infty} c(k)^2 < \infty$.

A sequence satisfying Assumption 3 is $c(k) = \frac{\eta}{k+1}$, for $\eta > 0$. The value of η in this chapter should be treated as a hyper-parameter that should be tuned, as it can improve the convergence of the proposed scheme. From our numerical examples, we have noticed that η in the interval $(0.1, 1)$ yields an acceptable performance.

Theorem 2. *Let $(x_i(k))_{k\in\mathbb{N}}$ be the sequences generated by Algorithm 1, for all $i = 1, \ldots, m$. Under Assumptions 1- 3, we have that for some minimiser $x^\star$ of* (3.1)*,*

$$\lim_{k\to\infty} \|x_i(k) - x^\star\|_2 = 0, \quad \forall\, i = 1, \ldots, m.$$

The proof of Theorem 2, as well as of Theorem 3 presented in 3.3.3, is based on some auxiliary technical results presented in Section 3.6.4.

Theorem 2 extends the result in [92] by allowing an agent to communicate subgradient information to neighbouring agents, a feature that, as illustrated in Section 3.4, can speed up practical convergence.

3.3.3 Convergence results for Algorithm 1: step sizes proportional to $\frac{1}{\sqrt{k}}$.

We now impose the following assumption on the step size $c(k)$.

Assumption 4. *The sequence $(c(k))_{k\in\mathbb{N}}$ used in Algorithm* (1) *is $c(k) = \frac{\eta}{\sqrt{k+1}}$, for some $\eta > 0$.*

Our convergence rate results build on the following related sequence generated by Algorithm 1,

$$\hat{x}_i(k+1) \coloneqq \frac{c(k+1)x_i(k+1) + S(k)\hat{x}_i(k)}{S(k+1)}, \tag{3.4}$$

where $S(k) = \sum_{r=1}^{k} c(r)$, and $(x_i(k))_{k\in\mathbb{N}}$, for all $i = 1,\dots,m$, are the sequences generated by Algorithm 1, with initial condition $\hat{x}_i(0) = x_i(0)$. By rewriting (3.4) as $\hat{x}_i(k) = \frac{1}{S(k)} \sum_{r=1}^{k} c(r)x_i(r)$, we can interpret (3.4) as a convex combination of past iterates.

Theorem 3. *Consider the running average defined in* (3.4)*. Under Assumptions 1, 2, and 4, we have that:*

i) For all $i, j = 1,\dots,m$, the sequence $(\|\hat{x}_i(k) - \hat{x}_j(k)\|)_{k\in\mathbb{N}}$ converges to zero at a rate $\mathcal{O}(\frac{\ln k}{\sqrt{k}})$.

ii) All accumulation points of the sequences $(\hat{x}_i(k))_{k\in\mathbb{N}}$, $i = 1,\dots,m$, are feasible.

iii) There exist $B_1, B_2 > 0$ such that

$$\left|\sum_{i=1}^{m} f_i(\hat{x}_i(k)) - f(x^\star)\right| \leq B_1\frac{1}{\sqrt{k}} + B_2\frac{\ln k}{\sqrt{k}}. \tag{3.5}$$

Note that Theorem 3 asserts convergence of the function value along the running average $\hat{x}_i(k)$, i.e., all limit points of $(\hat{x}_i(k))_{k\in\mathbb{N}}$ are optimal. However, the iterates might exhibit an oscillatory behaviour. If, for instance, function f is strongly convex, then Theorem 3 yields a convergence rate for the generated iterates. For the exact expression of B_1 and B_2, we refer the reader to Section 3.6.6. The absolute value in Theorem 3 is due to the fact that $\hat{x}_i(k)$ may not be necessarily feasible; however, item *ii*) in Theorem 3 implies that all accumulation points of $(\hat{x}_i(k))_{k\in\mathbb{N}}$, $i = 1,\dots,m$, are feasible. Item *i*) states the rate at which consensus is achieved for the sequences $(\hat{x}_i(k))_{k\in\mathbb{N}}$, $i = 1,\dots,m$.

It should be noted that the result of Theorem 3 further extends the work presented in [92] not only by allowing agents to communicate their (sub-)gradients, but by also unveiling how to (non trivially) adapt the proof line in that paper to come up with convergence results that recover traditional rates for distributed subgradient methods. Theorem 3 represents the first convergence rate result under the assumptions considered in this chapter.

rate

3.4 Numerical Examples

3.4.1 Problem instance of Section 3.2.2 – revisited

We revisit the two-agent problem in (3.3), for which the iterative scheme in (3.2) is not guaranteed to converge, and apply instead our algorithm. Note that the optimal solution of (3.3) is given by

$$x^\star = \mathcal{P}_{[0.5,1]^2}\left[-\frac{1}{8}Q^{-1}(q_1+q_2)\right] = \begin{bmatrix}0.5\\1\end{bmatrix}$$

where $\mathcal{P}_{[0.5,1]^2}[\cdot]$ represents the projection onto the feasible set of problem (3.3). Pictorially $x^\star$ is shown in Figure 3.1. To illustrate the convergence properties of Algorithm 1 we monitor the evolution of $\sqrt{\sum_{i=1}^{2}\|x_i(k)-x^\star\|_2^2}$, where $(x_i(k))_{k\in\mathbb{N}}$, $i=1,2$, are the iterates generated by Algorithm 1. We use $c(k)=\frac{1}{\sqrt{k+1}}$ similarly to [46], $A=\frac{1}{2}\mathbf{1}\mathbf{1}^\top$ and $x_i(0)=\hat{x}_i^\star$, where $\hat{x}_i^\star$, $i=1,2$, are defined in Section 3.2.2. Observe that our initial condition is the same as in Proposition 5. In contrast, as shown in Figure 3.2, the iterates generated by Algorithm 1 converge to the optimal solution of (3.3).

3.4.2 Example 2: robust linear regression

We consider the problem of estimating an unknown (but deterministic) vector $x\in\mathbb{R}^n$ from m noisy measurements y_i by means of the linear model

$$y_i = b_i^\top x + v_i, \quad i=1,\dots,m,$$

with $b_i\in\mathbb{R}^n$, and v_i are independent random variables drawn from a Laplacian distribution, that is, for each i the density of v_i is given by $h_{v_i}(z)=\frac{1}{2a}\exp^{-|z|/a}$, for all $z\in\mathbb{R}$. A common strategy is to impose a norm constraint of the form

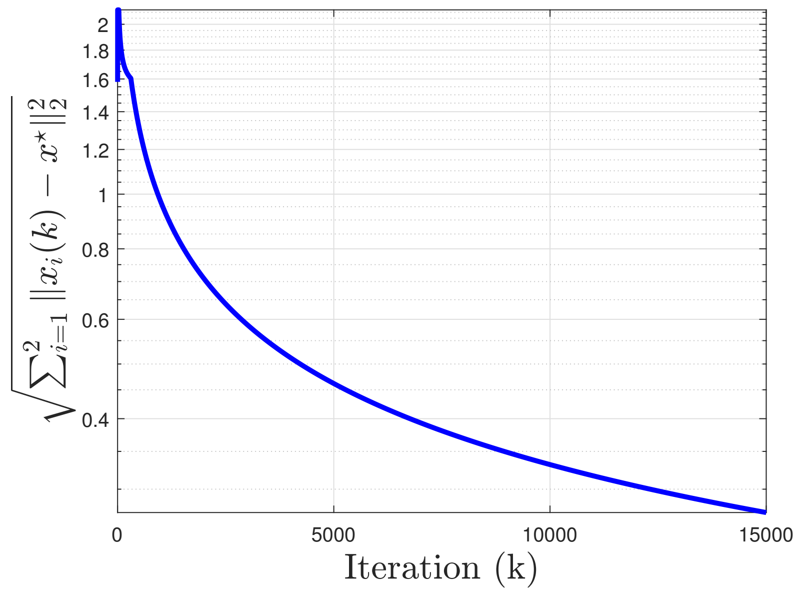


Figure 3.2: Evolution of $\sqrt{\sum_{i=1}^{2} \|x_i(k) - x^\star\|_2^2}$ for (3.3), where $(x_i(k))_{k\in\mathbb{N}}, i = 1, 2$, are the iterates generated by Algorithm 1.

$\|x\|_2 \leq c$, for some $c > 0$, to reflect some prior knowledge on the unknown vector x, and solve the second order conic program

$$\hat{x} \in \operatorname*{argmin}_{\|x\|_2 \leq c} \|y - Bx\|_1. \tag{3.6}$$

Typically, (3.6) is referred to as *robust regression* in the literature, as the ℓ_1-norm penalises relatively less outliers than other convex metrics (e.g., quadratic penalties). In our set-up, we consider the case where data are collected locally and agents are not willing to share their measurements with a central processing unit.

Observe that (3.6) has the format of (3.1) by setting $X_i = X = \{x \in \mathbb{R}^n : \|x\|_2 \leq 5\}$ and $f_i(x) = |y_i - b_i^\top x|$, $i = 1, \dots, m$. Moreover, the constraint sets X_i and the objective functions f_i, $i = 1, \dots, m$, trivially satisfy Assumption 1. Hence, we can apply the proposed scheme to obtain a solution to (3.6). We consider $m = 30$ and $n = 4$ and generate y independently from a standard Gaussian distribution, and matrix B from a uniform distribution with support $[0, 1]$.

We solve (3.6) in a distributed manner, and compare Algorithm 1 with the one proposed in [46] under four different network connectivity structures: *i*) complete network graph (which corresponds to the centralised version of the problem); *ii*) line network graph; *iii*) sparse network graph with sparsity degree $d = 0.3$; *iv*) sparse network graph with sparsity degree $d = 0.8$. We say that a network with m agents has a sparsity degree $d \in (0, 1)$ if the number of connections among the network nodes is given by dm^2, where m^2 indicates the number of connections of a complete graph.

We assess the performance of Algorithm 1 for each of the aforementioned networks in Figure 3.3. Solid lines correspond to Algorithm 1, whereas dashed lines correspond to the algorithm proposed in [46]. Different colours correspond to the different network connectivities. For each case, we monitor the evolution of $\frac{|\sum_{i=1}^{30} f_i(x_i(k)) - f^\star|}{f^\star}$, where $f^\star$ is the optimal value of (3.6). The proposed scheme exhibits similar and often favourable performance with the one in [46], in particular for cases where the underlying graph is not sparse. It should be noted, however, that Algorithm 1 possesses more general convergence properties, i.e., the proposed scheme is guaranteed to converge under non-identical local sets.

Note that due to the fact that Algorithm 1 requires two rounds of communication per iteration, the results presented in Figure 3.3 should be rescaled by a factor of two if we use communication rounds instead of the iteration index.

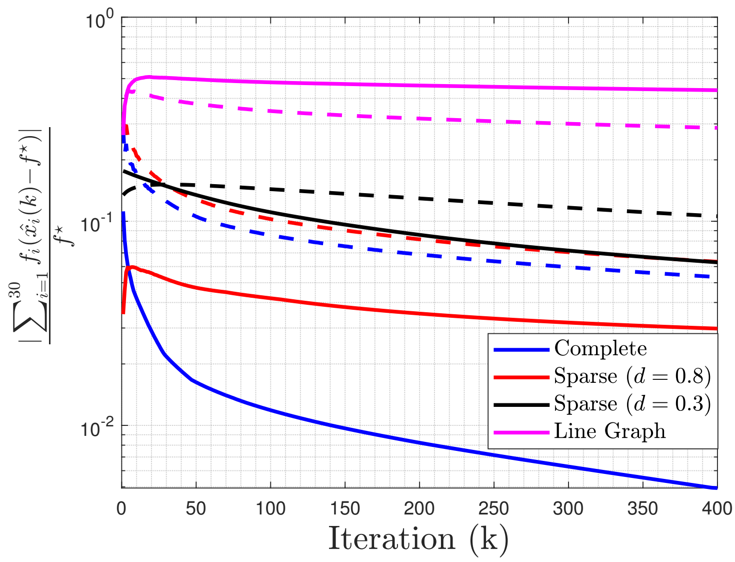


Figure 3.3: Evolution of $\frac{|\sum_{i=1}^{30} f_i(x_i(k)) - f^\star|}{f^\star}$ for Algorithm 1 (solid lines) and the one in [46] (dashed lines) when applied to the robust regression problem given by (3.6). The different colours correspond the different network connectivities.

3.4.3 Example 3: ℓ_2 linear regression with regularisation

In this example, we consider a variation of the regression problem where we assume v_i, $i = 1, \ldots, m$, to be independent and Gaussian, i.e., the density function is given by $h_{v_i}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, for all $z \in \mathbb{R}$, for all $i = 1, \ldots, m$, and we assume that x is sparse. A common relaxation of this problem is to choose the maximum likelihood estimator $\hat{x}$ such that

$$\hat{x} = \operatorname*{argmin}_{x \in X} \|y - Bx\|_2^2 + \lambda \|x\|_1, \tag{3.7}$$

where X can be interpreted as a set including prior beliefs, e.g., $\|x\|_2 \leq c$ or $\underline{x} \leq x \leq \bar{x}$ for some vectors $\underline{x}, \bar{x} \in \mathbb{R}^n$. The estimator $\hat{x}$ obtained by solving (3.7) depends on the value of the parameter λ. In fact, the larger the value of λ, the worse the performance is in terms of the error and the sparser the obtained solution is.

In this example, we aim to verify the performance of Algorithm 1 under step size choices $c(k) \propto \frac{1}{k+1}$ and a time-varying communication network. Similar to the previous example, the vector y is generated according to a standard normal distribution and matrix B from a uniform distribution on the interval $[0, 1]$. We assume $m > n$ and consider the case where agents possess private, local information, encoded by $X_i = [\underline{x}_i, \bar{x}_i]$ $i = 1, \ldots, m$, such that $X = \cap_{i=1}^m X_i = [\underline{x}, \bar{x}]$.

The algorithm presented in [46] does not necessarily converge in the set-up of problem (3.7), as we have different constraint sets per agent. We thus compare our algorithm against the one proposed in [92], which converges under similar conditions but does not leverage subgradient averaging. This allows us to assess the impact of averaging subgradients on practical convergence.

We now investigate the behaviour of the proposed algorithm in the presence of time-varying communication networks. To this end, we set $m = 300$ and $n = 10$, and generate four network configurations with different sparsity patterns, alternating cyclically among these. We also set $c(k) = \frac{0.2}{k+1}$ for both Algorithm 1 and the one in [92]. Figure 3.4 shows the evolution for the average distance to the optimal solution for Algorithm 1 (solid-red line) and the one in [92] (dashed-blue line). We

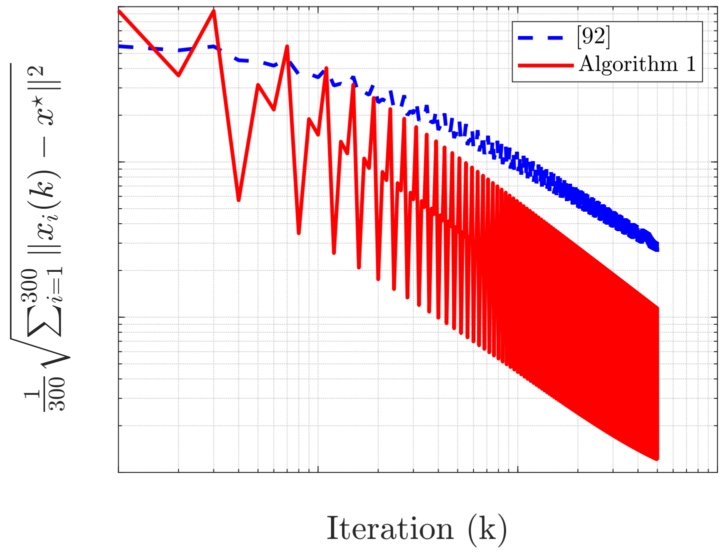


Figure 3.4: Evolution of the average distance to the optimal solution given by $\frac{\sqrt{\sum_{i=1}^{300} \|x_i(k)-x^\star\|_2^2}}{300}$ for Algorithm 1 (solid-red line) and that of [92] (dashed-blue line).

observe that Algorithm 1 consistently outperforms the one proposed in [92]; this is mainly due to the sub-gradient averaging step of Algorithm 1.

3.5 Conclusion

In this chapter we proposed a subgradient averaging algorithm for multi-agent optimisation problems involving non-differentiable objective functions and different constraint sets per agent. For this set-up we showed by means of a geometric construction that available schemes involving subgradient averaging cannot be used. For the proposed scheme we showed convergence of the algorithm iterates to some minimiser of a centralised problem counterpart. Moreover, we have also established a convergence rate under a particular choice for the underlying step size.

This concludes the study of the first challenge presented in Chapter 1, showing that scalability of current optimisation methods can be achieved by leveraging distributed computation exploiting a multi-agent setting in which agents communicate over a network. In the next chapter we focus on optimisation problems with integer decision variables.

3.6 Proofs of Chapter 3

3.6.1 Proof of Lemma 1

The proof of this Lemma relies on some concepts presented in Chapter 2. Let us start by proving item i). Consider the continuous mapping $\phi : \mathbb{R}^m \times \prod_{i=1}^m \mathbb{R}^n \to \mathbb{R}^n$, defined as $\phi(\gamma, x_1, \ldots, x_m) = \sum_{i=1}^m \gamma_i x_i$, where $\gamma = (\gamma_1, \ldots, \gamma_m)$ belongs to the simplex in $\mathbb{R}^m$, denoted by Γ. Consider $K = \phi(\Gamma, \prod_{i=1}^m X_i)$, and note that K is compact, as $\Gamma \times \prod_{i=1}^m X_i$ is compact and the image of a compact set under the continuous function is compact (see Proposition 1, Chapter 2). Moreover, note that by definition we have $K \subseteq \mathrm{conv}(\cup_{i=1}^m X_i)$, (due to Proposition 2, Chapter 2) as any element in K is a convex combination of elements in $\cup_{i=1}^m X_i$. To conclude the argument, we need to show that $\mathrm{conv}(\cup_{i=1}^m X_i) \subseteq K$. To this end, it suffices to show that K is a convex set, due to the fact that the convex hull is the smallest convex set containing a given set. Let $z, w \in K$, i.e., $z = \sum_{i=1}^m \gamma_i z_i$ and $w = \sum_{i=1}^m \beta_i w_i$, with $z_i, w_i \in X_i$, and $\gamma = (\gamma_1, \ldots, \gamma_m), \beta = (\beta_1, \ldots, \beta_m) \in \Gamma$. Fix an $\alpha \in (0,1)$, and note that $\alpha z + (1-\alpha) w = \sum_{i=1}^m (\alpha\gamma_i + (1-\alpha)\beta_i) x_i$, where $x_i = c_i z_i + (1-c_i) w_i \in A_i$, with $c_i = \frac{\alpha\gamma_i}{\alpha\gamma_i + (1-\alpha)\beta_i}$.

Since $x_i \in A_i$ due to convexity of A_i and $\alpha\gamma + (1-\alpha)\beta \in \Gamma$, we conclude that $\alpha z + (1-\alpha) w \in K$ for any $\alpha \in (0,1)$, thus showing that K is a convex set. This implies then that $K = \mathrm{conv}(\cup_{i=1}^m X_i)$ as we have established that $K \subseteq \mathrm{conv}(\cup_{i=1}^m X_i)$ and $\mathrm{conv}(\cup_{i=1}^m X_i) \subseteq K$. Since K was shown to be compact, we have that $\mathrm{conv}(\cup_{i=1}^m X_i)$ is also compact. This concludes the proof of item i). An alternative proof can be found at [10, Prop. 1.2.2]. The proof of item ii) follows from Proposition 5.4.2, p. 185, in [10], and is omitted. This concludes the proof of the lemma.

3.6.2 Sufficient condition for Assumption 1, item iii).

The goal of this subsection is to provide a sufficient condition for Assumption 1, item iii). The subsequent arguments can be found in standard optimisation books, such as [122, Theorem 24.7]; however we present here a more direct proof.

Assumption 5. *Let X_i, $i = 1, \ldots, m$, be the level sets of problem* (3.1) *and* dom$f$ *the domain of f. We suppose that:*

i) The distance between the set $\cup_{i=1}^m X_i$ and the complement of the interior of the closed and convex domain of f is strictly greater than zero, i.e.,

$$\mathrm{dist}\big(\cup_{i=1}^m X_i, \left(\mathrm{int}(\mathrm{dom} f)\right)^c \big) = \inf_{\substack{x \in \cup_{i=1}^m X_i, \\ y \in \left(\mathrm{int}(\mathrm{dom} f)\right)^c}} \|x - y\|_2^2 > 0.$$

ii) $X_i \subset \cap_{i=1}^m \mathrm{int}(\mathrm{dom} f_i)$ for each $i = 1, \ldots, m$.

As a consequence of Assumption 5, and since $\mathrm{dom} f = \cap_{i=1}^m \mathrm{dom} f_i$, $\mathrm{ri}(\mathrm{dom} f) = \cap_{i=1}^m \mathrm{ri}(\mathrm{dom} f_i)$ and $\mathrm{ri}(\mathrm{dom} f_i) \subset \mathrm{dom} f_i$ we have that the subdifferential $\partial f(x)$ is non empty for each $x \in \cap_{i=1}^m X_i$, due to item *ii*) of Assumption 5 and Proposition 4 in Chapter 2. Furthermore, $\partial f(x)$ is compact by [10, Proposition 5.4.1] since the affine hull of dom$f$ has dimension n due to Assumption 1, item *ii*). We use this fact to show that $\cup_{x \in \mathrm{conv}(\cup X_i)} \partial f(x)$ is a bounded set, that is, $\|g\|_2 \le L$, where $g \in \partial f(x)$ for any $x \in \cup_{i=1}^m X_i$. This result is formally stated in the next lemma.

Lemma 2. *Under Assumptions 1, items i) and ii) , and 5, we have that the set $\cup_{x \in \mathrm{conv}(\cup X_i)} \partial f(x)$ is non-empty and bounded.*

Proof. The proof of the lemma relies on Assumption 5, item *ii*), that is, $X_i \subset \cap_{j=1}^m \mathrm{ri}(\mathrm{dom} f_j)$, for all $i = 1, \ldots, m$. This implies that $\mathrm{conv}(\cup_{i=1}^m X_i) \subset \cap_{j=1}^m \mathrm{ri}(\mathrm{dom} f_j)$, as $\cap_{j=1}^m \mathrm{ri}(\mathrm{dom} f_j)$ is convex and contains $\cup_{i=1}^m X_i$. Suppose, by contradiction, that $\cup_{x \in \mathrm{conv}(\cup X_i)} \partial f(x)$ is unbounded. Then there exists a sequence $(x_k)_{k \in \mathbb{N}} \subset \mathrm{conv}(\cup_{i=1}^m X_i)$ such that $(g_k)_{k \in \mathbb{N}}$, with $g_k \in \partial f(x_k)$, satisfies $\|g_k\|_2 < \|g_{k+1}\|_2, \quad \forall k \in \mathbb{N}$.

Notice that $x_k \in \cap_{i=1}^m \mathrm{int}(\mathrm{dom} f_i)$ by Assumption 5, item *ii*). By item *i*) of Assumption 5, we can construct a sequence $(\beta_k)_{k \in \mathbb{N}}$ such that $x_k + \beta_k d_k \in \cap_{i=1}^m \mathrm{dom} f_i$. with $d_k = g_k / \|g_k\|_2$. Let $\beta = \inf_{k \in \mathbb{N}} \beta_k$ and notice that $\beta > 0$ (i.e., it is bounded away from zero) due to Assumption 5, item *i*). By the definition of g_k we have that

$$\frac{f(x_k + \beta d_k) - f(x_k)}{\beta} \ge \|g_k\|_2, \quad \forall\ k\ \in\ \mathbb{N}. \tag{3.8}$$

As inequality (3.8) is valid for all $k \in \mathbb{N}$, we take the limit superior on both sides to obtain

$$\limsup_{k\to\infty} \|g_k\|_2 \leq \limsup_{k\to\infty} \frac{f(x_i + \gamma d_k) - f(x_k)}{\gamma} < \infty, \tag{3.9}$$

where the right-hand side of (3.9) is finite as the sequences $(x_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ are bounded (notice that d_k is a normalised subgradient), and since f is continuous on its domain (f is convex). This establishes a contradiction, as we assumed $(g_k)_{k\in\mathbb{N}}$ were unbounded, thus concluding the proof of the Lemma.

□

3.6.3 Proof of Proposition 5

The proof is based on an induction argument.

Base case

We show that $z_i(1)^\top(\xi - \hat{x}_j^\star) \geq 0$, for all $\xi \in X_j$, for all $i, j = 1, 2$, and also that $x_i(1) = \hat{x}^\star$, for all $i = 1, 2$. Consider the inequalities

$$\nabla f_1(\hat{x}_1^\star)^\top(\xi - \hat{x}_i^\star) \geq 0, \quad \nabla f_2(\hat{x}_2^\star)^\top(\xi - \hat{x}_i^\star) \geq 0, \quad \forall \xi \in X_i, \quad i = 1, 2. \tag{3.10}$$

Fix $i = 1$. The first inequality in (3.10) holds due to optimality of $\hat{x}_1^\star$ [10]. To show the second inequality observe that $\nabla f_2(\hat{x}_2^\star) = [13.68, -3.94]^\top$, and that $\xi - \hat{x}_1^\star = [a_1, a_2]^\top$ with $a_1 \geq 0$ and $a_2 \leq 0$, for all $\xi \in X_1$. To see this latter fact, we can argue geometrically by inspecting Figure 3.1. Fix any point in the set X_1 and notice that vector point towards $\hat{x}_1^\star$ with base on this chosen point has the desired property.

Since $\nabla f_1(\hat{x}_1^\star) = [12, -4]^\top$, using a symmetric argument we show that

$$\nabla f_2(\hat{x}_2^\star)^\top(\xi - \hat{x}_2^\star) \geq 0, \quad \nabla f_1(\hat{x}_1^\star)^\top(\xi - \hat{x}_2^\star) \geq 0, \quad \forall \xi \in X_2. \tag{3.11}$$

By (3.2a), and under our choice for A,

$$z_i(1) = \frac{1}{2}\Big(\nabla f_1(\hat{x}_1^\star) + \nabla f_2(\hat{x}_2^\star)\Big) + \nabla f_i(\hat{x}_i^\star), \tag{3.12}$$

for $i = 1, 2$, hence inequalities (3.10) and (3.11) imply that $z_i(1)^\top(\xi - \hat{x}_j^\star) \geq 0$, $\forall \xi \in X_j$, for all $i, j = 1, 2$.

We will now prove that $x_i(1) = \hat{x}_i^\star$, for $i = 1, 2$. Fix $i = 1$. Since $z_1(1)^\top \xi + \frac{2}{c(k)} \|\xi\|_2^2$ is strictly convex, there is a unique point satisfying

$$\Big(z_1(1) + 2x_1(1)\Big)^\top (\xi - x_1(1)) \geq 0, \quad \forall \xi \in X_1, \tag{3.13}$$

where $(z_1(1) + 2x_1(1))$ is the gradient of the objective function in (3.2b) evaluated at $x_1(1)$, with $c(1) = 1$. Therefore, it suffices to show that

$$\Big(z_1(1) + 2\hat{x}_1^\star\Big)^\top (\xi - \hat{x}_1^\star) \geq 0, \quad \forall \xi \in X_1. \tag{3.14}$$

By substituting (3.3) into (3.12), we observe that $z_1(1) + 2\hat{x}_1^\star = [22.8414, -5.9708]^\top$, and due to the structure of $\xi - \hat{x}_1^\star$, (3.14) holds, thus proving that $x_1(1) = \hat{x}_1^\star$. A symmetric argument yields that $x_2(1) = x_2^\star$.

Induction hypothesis

Assume that $z_i(k)^\top(\xi - \hat{x}_j^\star) \geq 0$ for all $\xi \in X_j$, for $i, j = 1, 2$, and that $x_i(k) = x_i^\star$ for $i = 1, 2$. We aim to show that the aforementioned relations remain true for the step $k + 1$.

Proof for iteration $k + 1$

Fix $i = 1$. Following a similar reasoning with the base case, observe that $x_1(k+1) = x_1^\star$ if

$$\left[z_1(k+1) + \frac{2}{c(k)}\hat{x}_1^\star\right]^\top (\xi - \hat{x}_1^\star) \geq 0, \quad \forall \xi \in X_1. \tag{3.15}$$

As the sequence $(z_i(k))_{k \in \mathbb{N}}$ is generated by (3.2a), we propagate the dynamical system in (3.2a) by $k + 1$ steps to obtain

$$z_i(k+1) = \frac{1}{2}\Big(\nabla f_1(\hat{x}_1^\star) + \nabla f_2(\hat{x}_2^\star)\Big)(k+1) + \nabla f_1(\hat{x}_1^\star),$$

where we have used the fact that $A = \frac{1}{m}\mathbf{1}\mathbf{1}^\top$ and $c(k) = \frac{1}{\sqrt{k+1}}$. A sufficient condition for equation (3.15) to hold is that

$$\left[\frac{1}{2}\Big(\nabla f_1(\hat{x}_1^\star) + \nabla f_2(\hat{x}_2^\star)\Big)(k+1) + 2\hat{x}_1^\star\sqrt{k+1}\right]^\top (\xi - \hat{x}_1^\star) \geq 0, \quad \forall \xi \in X_1, \tag{3.16}$$

since $\nabla f_1(\hat{x}_1^\star)^\top(\xi - \hat{x}_1^\star) \geq 0$ by optimality of $\hat{x}_1^\star$. Recall that $(\xi - \hat{x}_1^\star) = [a_1, a_2]$ with $a_1 \geq 0$ and $a_2 \leq 0$ for all $\xi \in X_1$. To prove (3.16) we will show that the left-most vector in the same equation can be written as $[b_1, b_2]$ for some $b_1 \geq 0$ and $b_2 \leq 0$. To achieve this, notice that $k+1 \geq \sqrt{2}\sqrt{k+1}$, for all $k \geq 1$, and let e_i denote the unit vector with 1 in the i-th position, $i = 1, 2$. We then have that

$$e_1^\top \left[\frac{1}{2}\Big(\nabla f_1(\hat{x}_1^\star) + \nabla f_2(\hat{x}_2^\star)\Big)\right](k+1) \\ \geq e_1^\top \left[\frac{\sqrt{2}}{2}\Big(\nabla f_1(\hat{x}_1^\star) + \nabla f_2(\hat{x}_2^\star)\Big)\right]\sqrt{k+1}, \tag{3.17}$$

and

$$2e_2^\top \hat{x}_1^\star \sqrt{k+1} \leq \sqrt{2} e_2^\top \hat{x}_1^\star (k+1), \tag{3.18}$$

since the first component of the averaged gradient and the second component of $\hat{x}_1^\star$ are both positive. Therefore, for all $k \in \mathbb{N}$,

$$b_1 \geq 16.1604\sqrt{k+1} > 0, \quad b_2 \leq -2.5566(k+1) < 0. \tag{3.19}$$

Inequalities (3.17), (3.18) and (3.19), together with the structure of $\xi - \hat{x}_1^*$, imply that (3.16) holds, so we can conclude that $x_1(k+1) = \hat{x}_1^\star$. A symmetric argument shows that $x_2(k+1) = \hat{x}_2^\star$.

To complete the proof it remains to show that $z_i(k+1)^\top(\xi - \hat{x}_j^\star) \geq 0$ for all $\xi \in X_j$, for all $i, j = 1, 2$, where $z_i(k+1) = \frac{1}{2}\Big(z_1(k) + z_2(k)\Big) + \nabla f_i(x_i(k))$, due to (3.2a) and our choice for A. By our induction hypothesis, $z_i(k)(\xi - \hat{x}_j^\star) \geq 0$, for all $i, j = 1, 2$, hence it suffices to show that $\nabla f_i(x_i(k))^\top(\xi - \hat{x}_j^\star) \geq 0$, $\forall \xi \in X_j$, $\forall i = 1, 2$. Since $x_i(k) = \hat{x}_i^\star$ for $i = 1, 2$, due to our induction hypothesis, the claim follows from (3.10) and (3.11), thus concluding the proof.

3.6.4 Auxiliary Lemmas for the proofs of Theorem 2 and 3.

Let

$$v(k) = \frac{1}{m}\sum_{i=1}^{m} x_i(k), \tag{3.20}$$

be the average of the agents' estimates at time k. Since this quantity might not necessarily belong to the feasible set $\cap_{i=1}^m X_i$, we define

$$\bar{v}(k) = \frac{\rho}{\epsilon(k) + \rho} v(k) + \frac{\epsilon(k)}{\epsilon(k) + \rho}\bar{x}, \tag{3.21}$$

where $\bar{x}$ is a point in the interior of the feasible set (which is non-empty by Assumption 1, item *ii)*), $\rho > 0$ is such that the 2-norm ball of centre $\bar{x}$ and radius ρ is contained in $\cap_{i=1}^m X_i$, and $\epsilon(k) = \sum_{i=1}^m \mathrm{dist}(v(k), X_i)$. As shown in [103], $\bar{v}(k) \in \cap_{i=1}^m X_i$, for all $k \in \mathbb{N}$. We also define $e_i(k+1) = x_i(k+1) - z_i(k)$, and note that the z_i-update in Algorithm 1 can be written as

$$x_i(k+1) = \sum_{j=1}^m a_{ji}(k) x_j(k) + e_i(k+1). \tag{3.22}$$

Lemma 3. *The following relations hold.*

i) Let $(x_i(k))_{k\in\mathbb{N}}$, $i = 1, \ldots, m$, be the sequences generated by Algorithm 1, and $(v(k))_{k\in\mathbb{N}}$ and $(\bar{v}(k))_{k\in\mathbb{N}}$ defined by (3.20) *and* (3.21)*, respectively. Under Assumption 1, we have that for all $k \geq 0$,*

$$\sum_{i=1}^m \|x_i(k+1) - \bar{v}(k)\|_2 \leq \mu \sum_{i=1}^m \|x_i(k) - v(k)\|_2,$$

where $\mu = \frac{2}{\rho} mD + 1$, and D is the diameter of the set $\cup_{i=1}^m X_i$ (which is well-defined by Lemma 1, item i)).

ii) Let $(x_i(k))_{k\in\mathbb{N}}$, $i = 1, \ldots, m$, and $(v(k))_{k\in\mathbb{N}}$ be as in item i). Under Assumption 2, we have that for all $i = 1, \ldots, m$, for all $k \geq 0$,

$$\|x_i(k+1) - v(k+1)\|_2 \leq \lambda q^k \sum_{j=1}^m \|x_j(0)\|_2 + \|e_i(k+1)\|_2 + \sum_{r=0}^{k-1} \lambda q^{k-r-1} \sum_{j=1}^m \|e_j(r+1)\|_2 + \frac{1}{m} \sum_{j=1}^m \|e_j(k+1)\|_2,$$

where $\lambda = 2(1+\eta^{-(m-1)T})/(1-\eta^{(m-1)T}) \in \mathbb{R}_+$ and $q = (1-\eta^{(m-1)T})^{\frac{1}{(m-1)T}} \in (0,1)$.

iii) Given a non-increasing and non-negative sequence $(c(k))_{k\in\mathbb{N}}$, and a scalar $\bar{L} > 0$, we have that

$$2\bar{L} \sum_{k=0}^N c(k) \sum_{i=1}^m \|x_i(k+1) - \bar{v}(k+1)\|_2 < \beta_1 \sum_{k=0}^N \sum_{i=1}^m \|e_i(k+1)\|_2^2 + \beta_2 \sum_{k=0}^N c(k)^2 + \beta_3,$$

where $\beta_1 \in (0,1)$, and β_2 and β_3 are positive constants.

Proof. The proof of item *i*) is presented in [92, Lemma 1]. For item *ii*), see [92, Lemma 2]. Finally, the proof of item *iii*) follows the line of [92, Lemma 3]. We highlight that constants β_1, β_2, and β_3 depend on the parameter μ defined in item *i*). Their exact expressions can be found in [92], equation (35). □

Observe that the values of λ and q in Lemma 3, item *ii*), depend on the parameter T that characterises the uniform bound in Assumption 2, item *i*); and on η, the lower bound for the elements of $A(k)$, Assumption 2, item *ii*). The following lemma is instrumental for the proof of Theorem 3. In particular, Lemma 4, item *ii*), constitutes a non-trivial extension of the result in [92], allowing some sequences to be iteration-varying.

Lemma 4. *Let* $(x_i(k))_{k\in\mathbb{N}}, (z_i(k))_{k\in\mathbb{N}}$ *and* $(d_i(k))_{k\in\mathbb{N}}$, $i = 1, \ldots, m$, *be the sequences generated by Algorithm 1, and* $x^\star$ *by any optimal solution of* (3.1)*. Under Assumptions 1 and 2, we have that:*

i) For all $k \in \mathbb{N}$,

$$2c(k)\sum_{i=1}^{m} d_i(k)^\top (x_i(k+1) - x^\star) + \sum_{i=1}^{m} \|e_i(k+1)\|_2^2 + \sum_{i=1}^{m} \|x_i(k+1) - x^\star\|_2^2 \leq \sum_{i=1}^{m} \|x_i(k) - x^\star\|_2^2. \quad (3.23)$$

ii) For any $\beta_1 \in (0,1)$*, there exist sequences* $(\alpha_1(k))_{k\in\mathbb{N}}$ *and* $(\alpha_2(k))_{k\in\mathbb{N}}$ *such that, for all* $k \in \mathbb{N}$, $\alpha_1(k) \in (0,1)$, $\alpha_2(k) \in (0,1)$, $1 - \beta_1 - \alpha_1(k) - \alpha_2(k) \geq 0$ *and*

$$\begin{aligned}2\sum_{k=0}^{N} c(k)\sum_{i=1}^{m}(f_i(\bar{v}(k+1)) - f_i(x^\star)) + \sum_{k=0}^{N}\sum_{i=1}^{m} \|x_i(k+1) - x^\star\|_2^2 \\ + \sum_{k=0}^{N}(1 - \alpha_1(k) - \alpha_2(k) - \beta_1)\sum_{i=1}^{m} \|e_i(k+1)\|_2^2 \leq \sum_{k=0}^{N}\sum_{i=1}^{m} \|x_i(k) - x^\star\|_2^2 \\ + \sum_{k=0}^{N}\Big(mL^2\frac{\alpha_1(k) + \alpha_2(k)}{\alpha_1(k)\alpha_2(k)} + \beta_2\Big)c(k)^2 + \beta_3.\end{aligned} \quad (3.24)$$

Proof. Item i): Fix any $i \in \{1, \dots, m\}$ and consider the sequence $(x_i(k))_{k\in\mathbb{N}}$. By optimality of $x_i(k+1)$ (see Algorithm 1), for any $\xi \in X_i$,

$$\begin{aligned} d_i(k)^\top x_i(k+1) - \frac{1}{c(k)}(z_i(k) - x_i(k+1))^\top x_i(k+1) \\ \leq d_i(k)^\top \xi - \frac{1}{c(k)}(z_i(k) - x_i(k+1))^\top \xi, \end{aligned} \tag{3.25}$$

where $d_i(k) - \frac{1}{c(k)}(z_i(k) - x_i(k+1))$ constitutes the gradient of the objective function in the x_i-update of Algorithm 1, evaluated at $x_i(k+1)$. Fix any optimal solution of (3.1), $x^\star \in \cap_{i=1}^m X_i$, and consider the following identity

$$\begin{aligned} &- \frac{1}{c(k)}(z_i(k) - x_i(k+1))^\top (x_i(k+1) - x^\star) \\ &= \frac{1}{2c(k)}\|x_i(k+1) - z_i(k)\|_2^2 + \frac{1}{2c(k)}\|x_i(k+1) - x^\star\|_2^2 - \frac{1}{2c(k)}\|z_i(k) - x^\star\|_2^2. \end{aligned} \tag{3.26}$$

Combining (3.26) and (3.25) with $\xi = x^\star$, we obtain

$$\begin{aligned} &d_i(k)^\top x_i(k+1) + \frac{1}{2c(k)}\|x_i(k+1) - z_i(k)\|_2^2 + \frac{1}{2c(k)}\|x_i(k+1) - x^\star\|_2^2 \\ &\leq d_i(k)^\top x^\star + \frac{1}{2c(k)}\|z_i(k) - x^\star\|_2^2 \\ &\leq d_i(k)^\top x^\star + \frac{1}{2c(k)} \sum_{j=1}^m a_{ji}(k)\|x_j(k) - x^\star\|_2^2, \end{aligned} \tag{3.27}$$

where the last inequality follows from double stochasticity of $A(k)$ and convexity of $\|\cdot\|^2$. We now multiply both sides of (3.27) by $2c(k)$ and sum the result for all $i = 1, \dots, m$, to obtain

$$\begin{aligned} 2c(k)\sum_{i=1}^m d_i(k)^\top x_i(k+1) + \sum_{i=1}^m \|x_i(k+1) - z_i(k)\|_2^2 + \sum_{i=1}^m \|x_i(k+1) - x^\star\|_2^2 \\ \leq 2c(k)\sum_{i=1}^m d_i(k)^\top x^\star + \sum_{i=1}^m \|x_i(k) - x^\star\|_2^2, \end{aligned} \tag{3.28}$$

where $\sum_{i=1}^m \sum_{j=1}^m a_{ji}(k)\|x_j(k) - x^\star\|_2^2 = \sum_{i=1}^m \|x_i(k) - x^\star\|_2^2$ by exchanging the order of summation, and due to double stochasticity of $A(k)$. The result follows from (3.28) by recalling that $e(k+1) = x_i(k+1) - z_i(k)$ and moving the first term in the right-hand side of (3.28) to the left one. This concludes the proof of item i).

Item *ii*): Consider the first term in the left-hand side of (3.23), and rewrite it as

$$2c(k)\sum_{i=1}^{m} d_i(k)^\top (x_i(k+1)-x^\star) = 2c(k)\sum_{i=1}^{m} d_i(k)^\top (x_i(k+1)-\bar{v}(k+1)) + 2c(k)\sum_{i=1}^{m} d_i(k)^\top (\bar{v}(k+1)-x^\star) \quad (3.29)$$

by adding and subtracting $\bar{v}(k+1)$. We next consider the terms in the right hand-side of (3.29) separately. First, observe that

$$2c(k)\sum_{i=1}^{m} d_i(k)^\top (x_i(k+1)-\bar{v}(k+1)) \geq -2c(k)L\sum_{i=1}^{m} \|x_i(k+1)-\bar{v}(k+1)\|_2, \quad (3.30)$$

by the Cauchy-Schuwarz inequality, where $L = \max_{\xi\in\cup_{i=1}^m X_j} \|g_j(\xi)\|_2$, which is well-defined due to Lemma 1. Using the definition of $d_i(k)$ – see Algorithm 1 – into the second term in the right-hand side of (3.29), we then have that (via double stochasticity of A)

$$2c(k)\sum_{i=1}^{m} d_i(k)^\top (\bar{v}(k+1)-x^\star) = 2c(k)\sum_{i=1}^{m} g_i(z_i(k))^\top (\bar{v}(k+1)-x^\star). \quad (3.31)$$

Moreover, by adding and subtracting $x_i(k+1)$ and $z_i(k)$ for all $i = 1,\ldots,m$, into the right-hand side of (3.31) we obtain

$$2c(k)\sum_{i=1}^{m} g_i(z_i(k))^\top (\bar{v}(k+1)-x^\star) = 2c(k)\sum_{i=1}^{m} g_i(z_i(k))^\top (\bar{v}(k+1)-x_i(k+1)) +2c(k)\sum_{i=1}^{m} g_i(z_i(k))^\top (x_i(k+1)-z_i(k)) + 2c(k)\sum_{i=1}^{m} g_i(z_i(k))^\top (z_i(k)-x^\star). \quad (3.32)$$

Consider now the right-hand side of (3.32). The left-most term can be lower-bounded as

$$2c(k)\sum_{i=1}^{m} g_i(z_i(k))^\top (\bar{v}(k+1)-x_i(k+1)) \geq -2c(k)L\sum_{i=1}^{m} \|\bar{v}(k+1))-x_i(k+1)\|_2, \quad (3.33)$$

by the Cauchy-Schuwarz inequality. As for the middle term, we have that

$$2c(k)\sum_{i=1}^{m} g_i(z_i(k))^\top (x_i(k+1)-z_i(k)) \geq -2c(k)L\sum_{i=1}^{m} \|e_i(k+1)\|_2 \geq -\alpha_1(k)\sum_{i=1}^{m} \|e_i(k+1)\|_2^2 - m\frac{L^2}{\alpha_1(k)}c(k)^2 \quad (3.34)$$

where the first inequality follows from the Cauchy-Schuwarz inequality and the definition $e_i(k)$ in (3.22). For the second inequality, we employed the relation $2xy \leq x^2 + y^2$ with $x = \frac{L}{\sqrt{\alpha_1(k)}}c(k)$ and $y = \sqrt{\alpha_1(k)}\|e_i(k+1)\|_2$ for some $\alpha_1(k) \in (0,1)$, $k \in \mathbb{N}$.

Similarly, the right-most term of (3.32) can be manipulated to yield

$$
\begin{aligned}
2c(k)\sum_{i=1}^{m} g_i(z_i(k))^\top(z_i(k) - x^\star) &\geq 2c(k)\sum_{i=1}^{m}\Big(f_i(z_i(k)) - f_i(x^\star)\Big) \\
= 2c(k)\sum_{i=1}^{m}\Big(f_i(z_i(k)) - f_i(\bar{v}(k+1))\Big) &+ 2c(k)\sum_{i=1}^{m}\Big(f_i(\bar{v}(k+1)) - f_i(x^\star)\Big) \qquad (3.35)
\end{aligned}
$$

where the inequality follows from the definition of the subgradient for a convex function, and the equality by adding and subtracting $f_i(\bar{v}(k+1))$. The first term in the right-hand side of (3.35) can be lower bounded as

$$
\begin{aligned}
2c(k)\sum_{i=1}^{m}\Big(f_i(z_i(k)) - f_i(\bar{v}(k+1))\Big) &\geq -2c(k)L\sum_{i=1}^{m}\|z_i(k) - \bar{v}(k+1)\|_2 \\
\geq -2c(k)L\left(\sum_{i=1}^{m}\left(\|e_i(k+1)\|_2 + \|x_i(k+1) - \bar{v}(k+1)\|_2\right)\right) & \\
\geq -\alpha_2(k)\sum_{i=1}^{m}\|e_i(k+1)\|_2^2 - m\frac{L^2}{\alpha_2(k)}c(k)^2 & \\
- 2c(k)L\sum_{i=1}^{m}\|x_i(k+1) - \bar{v}(k+1)\|_2 & \qquad (3.36)
\end{aligned}
$$

where the first inequality follows from the relation $x \geq -|x|$, for all $x \in \mathbb{R}$, and from item *iii*) of Lemma 1, and the second inequality by adding and subtracting $x_i(k+1)$, for all $i = 1, \ldots, m$, and then using triangle inequality. The last inequality follows from $2xy \leq x^2 + y^2$ with $x = \frac{L}{\sqrt{\alpha_2(k)}}c(k)$ and $y = \sqrt{\alpha_2(k)}\|e_i(k+1)\|_2$ for some $\alpha_2(k) \in (0,1)$, $k \in \mathbb{N}$. Substituting (3.36) into (3.35)

$$
\begin{aligned}
2c(k)\sum_{i=1}^{m} g_i(z_i(k))^\top(z_i(k) - x^\star) &\geq -\alpha_2(k)\sum_{i=1}^{m}\|e_i(k+1)\|_2^2 - m\frac{L^2}{\alpha_2(k)}c(k)^2 \\
- 2c(k)L\sum_{i=1}^{m}\|x_i(k+1) - \bar{v}(k+1)\|_2 &+ 2c(k)\sum_{i=1}^{m}\Big(f_i(\bar{v}(k+1)) - f_i(x^\star)\Big). \qquad (3.37)
\end{aligned}
$$

Substituting (3.29), (3.30), (3.33), (3.34), (3.37) into (3.23)

$$2c(k)\sum_{i=1}^{m}(f_i(\bar{v}(k+1))-f_i(x^\star))+\Big(1-\alpha_1(k)-\alpha_2(k)\Big)\sum_{i=1}^{m}\|e_i(k+1)\|_2^2$$
$$\sum_{i=1}^{m}\|x_i(k+1)-x^\star\|_2^2+\leq\sum_{i=1}^{m}\|x_i(k)-x^\star\|_2^2+mL^2\Big(\frac{\alpha_1(k)+\alpha_2(k)}{\alpha_1(k)\alpha_2(k)}\Big)c(k)^2$$
$$+6c(k)L\sum_{i=1}^{m}\|x_i(k+1)-\bar{v}(k+1)\|_2. \qquad (3.38)$$

Summing (3.38) from $k=0$ to $k=N$, and using Lemma 4, item *iii*), with $\bar{L}=3L$, the desired inequality (3.24) follows. This concludes the proof of item *ii*). □

Note that for any $\beta_1\in(0,1)$, the sequences $(\alpha_1(k))_{k\in\mathbb{N}}$ and $(\alpha_2(k))_{k\in\mathbb{N}}$ can be chosen to guarantee that $1-\alpha_1(k)-\alpha_2(k)-\beta_1\geq 0$ for all $k\in\mathbb{N}$. For instance, one particular choice is $\alpha_1(k)=\alpha_2(k)=\alpha$ with $1-\beta_1-2\alpha>0$. Three immediate consequences of Lemma 4 are presented in the following proposition.

Proposition 6. *Consider Assumptions 1–3. The following statements hold*

i) We have that $\sum_{k=0}^{\infty}\sum_{i=1}^{m}\|e_i(k)\|_2^2<\infty$*;*

ii) For all $i=1,\dots,m$*, we have that* $\lim_{k\to\infty}\|e_i(k)\|_2=0$*;*

iii) For all $i=1,\dots,m$*,* $\lim_{k\to\infty}\|x_i(k)-v(k)\|_2=0$.

Proof. Item *i*): Consider Lemma 4, item *ii*). Note that $\sum_{k=0}^{N}\sum_{i=1}^{m}\|x_i(k+1)-x^\star\|_2$ and $\sum_{k=0}^{N}\sum_{i=1}^{m}\|x_i(k)-x^\star\|_2$ form a telescopic series, so they can be replaced by $\sum_{i=1}^{m}\|x_i(N+1)-x^\star\|_2$ and $\sum_{i=1}^{m}\|x_i(0)-x^\star\|_2$, respectively. Let $\beta_1\in(0,1)$, choose $\alpha_1(k)=\alpha_2(k)=\alpha$ so that $1-2\alpha-\beta_1>0$. Observe that $\sum_{i=1}^{m}(f_i(\bar{v}(k+1))-f_i(x^\star))\geq 0$ for all $k\in\mathbb{N}$, due to optimality of $x^\star$, so this term can be dropped from (3.24). Besides, we can also drop the term $\sum_{i=1}^{m}\|x_i(N+1)-x^\star\|_2^2\geq 0$ since it is non-negative and appears in the left-hand side of (3.24). This yields

$$(1-2\alpha-\beta_1)\sum_{k=0}^{N}\sum_{i=1}^{m}\|e_i(k+1)\|_2^2\leq\sum_{i=1}^{m}\|x_i(0)-x^\star\|_2^2$$
$$+\left(mL^2\frac{2}{\alpha}+\beta_2\right)\sum_{k=0}^{N}c(k)^2+\beta_3.$$

Letting $N \to \infty$, we conclude that $\sum_{k=0}^{\infty} \sum_{i=1}^{m} \|e_i(k)\|_2^2$ is finite since the sequence $(c(k))_{k\in\mathbb{N}}$ is square-summable under Assumption 3 and the feasible set is compact. This concludes the proof of item *i*).

Item *ii*): Follows directly from item *i*).

Item *iii*): This proof follows directly from the arguments presented in [92, Proposition 3], and is omitted. □

3.6.5 Proof of Theorem 2

We are now in a position to prove Theorem 2. To this end, we use the inequality (3.38) and leverage on Lemma 3.4 in [12] to establish convergence of the sequences $(\|x_i(k) - x^\star\|_2)_{k\in\mathbb{N}}$, $i = 1, \dots, m$, to zero for some minimiser $x^\star$ of (3.1). We first present Lemma 3.4 in [12].

Lemma 5 ([12])**.** *Consider non-negative scalar sequences* $(\ell(k))_{k\in\mathbb{N}}$, $(u(k))_{k\in\mathbb{N}}$ *and* $(\zeta(k))_{k\in\mathbb{N}}$ *that satisfy the recursion* $\ell(k+1) \leq \ell(k) - u(k) + \zeta(k)$*. If* $\sum_{k=0}^{\infty} \zeta(k) < \infty$*, then the sequence* $(\ell(k))_{k\in\mathbb{N}}$ *converges and the sequence* $(u(k))_{k\in\mathbb{N}}$ *is summable.*

Consider inequality (3.38), and choose $\alpha_1(k), \alpha_2(k)$ and β_1 as in the proof of Proposition 6 item *i*). We now drop the term involving $(1-2\alpha)\sum_{i=1}^{m} \|e_i(k+1)\|_2^2$ as it appears on the left-hand side of the inequality and is non-negative so that we obtain

$$\sum_{i=1}^{m} \|x_i(k+1) - x^\star\|_2^2 \leq \sum_{i=1}^{m} \|x_i(k) - x^\star\|_2^2 - 2c(k) \sum_{i=1}^{m} (f_i(\bar{v}(k+1)) - f_i(x^\star)) + \frac{2mL^2}{\alpha} c(k)^2 + 6c(k)L \sum_{i=1}^{m} \|x_i(k+1) - \bar{v}(k+1)\|_2. \tag{3.39}$$

With reference to Lemma 5 and considering inequality (3.39), we set $\ell(k) = \sum_{i=1}^{m} \|x_i(k) - x^\star\|_2^2$, and

$$\begin{aligned} \zeta(k) &= \frac{2mL^2}{\alpha} c(k)^2 + 6c(k)L \sum_{i=1}^{m} \|x_i(k+1) - \bar{v}(k+1)\|_2, \\ u(k) &= 2c(k)\Big(f(\bar{v}(k+1)) - f(x^\star)\Big). \end{aligned} \tag{3.40}$$

By Lemma 3, item iii), with $\bar{L} = 3L$, and by Proposition 6, item i), it follows that $6L \sum_{k=1}^{\infty} c(k) \sum_{i=1}^{m} \|x_i(k+1) - \bar{v}(k+1)\| < \infty$, hence, $\sum_{k=1}^{\infty} \zeta(k) < \infty$, as $c(k)$

is square-summable due to Assumption 3, which implies that the assumptions of Lemma 5 hold.

Therefore, we have that the sequence $(\sum_{i=1}^m \|x_i(k) - x^\star\|_2^2)_{k\in\mathbb{N}}$ converges, which implies that $(\sum_i \|x_i(k)-x^\star\|_2)_{k\in\mathbb{N}}$ also converges. To see this, note that, by continuity of the square-root function, $(\sum_{i=1}^m \|x_i(k) - x^\star\|_2^2)_{k\in\mathbb{N}}$ being a convergent sequence implies that $(\|X(k) - x^\star \otimes \mathbf{1}^\top\|_F)_{k\in\mathbb{N}}$ also converges, where, for a fixed $k \in \mathbb{N}$, $X(k)$ is a $n \times m$ matrix whose i-th column is given by $x_i(k)$, and $\otimes$ represents the Kronecker product. Moreover, note that the set of $n \times m$ matrices can be equipped with the norm $\sum_{i=1}^m \|x_i\|_2$, where x_i, $i = 1, \dots, m$, is the i-th column of a generic element $X \in \mathbb{R}^{n\times m}$. Since all norms in finite-dimensional spaces are equivalent, we conclude that the sequence $(\sum_{i=1}^m \|x_i(k) - x^\star\|_2)_{k\in\mathbb{N}}$ also converges. An alternative but more tedious justification of this argument can be found in [92].

By Lemma 5, we also have that $\sum_{k=1}^\infty c(k)\Big(f(\bar{v}(k+1)) - f(x^\star)\Big) < \infty$. The latter implies that $\liminf_{k\to\infty}(f(\bar{v}(k+1)) - f(x^\star)) = 0$. Therefore, there exists a subsequence of $(f(\bar{v}(k+1)) - f(x^\star))_{k\in\mathbb{N}}$ that converges to zero. Since the function $f(x)$ is continuous (by convexity) there exists some minimizer $x^\star$ such that a subsequence of $(\|\bar{v}(k) - x^\star\|_2)_{k\in\mathbb{N}}$ converges to zero. Moreover, we obtain $\sum_{i=1}^m \|x_i(k) - x^\star\|_2 \leq \sum_{i=1}^m \|\bar{v}(k) - x^\star\|_2 + \mu \sum_{i=1}^m \|x_i(k) - v(k)\|_2$. by adding and subtracting $\bar{v}(k)$, then applying triangle inequality and invoking Lemma 3, item i).

Note that $(\|\bar{v}(k) - x^\star\|_2)_{k\in\mathbb{N}}$ converges to zero across a subsequence and the sequence $(\sum_{i=1}^m \|x_i(k)-v(k)\|_2)_{k\in\mathbb{N}}$ converges to zero (due to Proposition 6, item iii)) hence we can find a subsequence of $(\sum_{i=1}^m \|x_i(k) - x^\star\|_2)_{k\in\mathbb{N}}$ that converges to zero. However, we have shown by means of Lemma 5 that the sequence $(\sum_{i=1}^m \|x_i(k) - x^\star\|_2)_{k\in\mathbb{N}}$ converges; as a result it should converge to zero since every Cauchy sequence has a unique limit point. To conclude the proof, note that, for all $k \in \mathbb{N}$ and for all $j = 1, \dots, m$, $\|x_j(k) - x^\star\|_2 \leq \sum_{i=1}^m \|x_i(k) - x^\star\|_2$, so we conclude that the sequences $(\|x_j(k) - x^\star\|_2)_{k\in\mathbb{N}}$, $j = 1, \dots, m$, converge to zero. This concludes the proof.

3.6.6 Proof of Theorem 3

Consider Assumption 4. We drop the constant η for simplicity of exposition, but general choices $\frac{\eta}{\sqrt{k+1}}$, $\eta > 0$, are also applicable. Let $(\hat{v}(k))_{k\in\mathbb{N}}$ be the running average sequence associated with $(\bar{v}(k))_{k\in\mathbb{N}}$ (definition is analogous to $(\hat{x}_i(k))_{k\in\mathbb{N}}$ in (3.4)). Note that since $\cap_{i=1}^m X_i$ is assumed to be convex, we have that $\hat{v}(k)$ is feasible for all $k \in \mathbb{N}$ (see also the discussion below (3.21)). We have that

$$\left|\sum_{i=1}^m f_i(\hat{x}_i(k+1)) - f(x^\star)\right| \leq f(\hat{v}(k+1)) - f(x^\star) + L\sum_{i=1}^m \|\hat{x}_i(k+1) - \hat{v}(k+1)\|_2, \tag{3.41}$$

which follows from triangle inequality and Lemma 1, item *iii*). Note that the first term in the right-hand side of (3.41) does not involve an absolute value due to feasibility of the sequence $(\hat{v}(k))_{k\in\mathbb{N}}$, which in turn implies that $f(\hat{v}(k+1)) \geq f(x^\star)$.

To facilitate subsequent statements, we change the notation in Lemma 4, item *ii*), by replacing k by r, and N by k. The inequality with this modified notation is repeated here for clarity. Indeed, we have that for all $k \in \mathbb{N}$

$$2\sum_{r=0}^k c(r)\sum_{i=1}^m (f_i(\bar{v}(r+1)) - f_i(x^\star)) + \sum_{r=0}^k (1-\alpha_1(r)-\alpha_2(r)-\beta_1)\sum_{i=1}^m \|e_i(r+1)\|_2^2 + \sum_{r=0}^k\sum_{i=1}^m \|x_i(r+1)-x^\star\|_2^2 \leq \sum_{r=0}^k\sum_{i=1}^m \|x_i(r)-x^\star\|_2^2 + \sum_{r=0}^k \Big(mL^2\frac{\alpha_1(r)+\alpha_2(r)}{\alpha_1(r)\alpha_2(r)} + \beta_2\Big)c(r)^2 + \beta_3, \tag{3.42}$$

where $(\alpha_1(r))_{r\in\mathbb{N}}$ and $(\alpha_2(r))_{r\in\mathbb{N}}$ are sequences such that $1-\beta_1-\alpha_1(r)-\alpha_2(r) \geq 0$ for all $r \in \mathbb{N}$.

The proofs of items *i*), *ii*) and *iii*) of Theorem 3 are intertwined and will be composed into two parts: we first assume that there exist constants $d_1, d_2, d_3, d_4 > 0$ such that (3.43) and (3.44) bellow are satisfied, and on this basis prove the claims of the theorem; we then return to (3.43) and (3.44), and prove the existence of such constants. To this end, consider

$$f(\hat{v}(k+1)) - f(x^\star) \leq d_1 \frac{1}{S(k+1)} + d_2 \frac{\sum_{r=0}^{k} c(r)^2}{S(k+1)} \tag{3.43}$$

$$L \sum_{i=1}^{m} \|\hat{x}_i(k+1) - \hat{v}(k+1)\|_2 \leq \frac{d_3}{S(k+1)} + d_4 \frac{\sum_{r=0}^{k} c(r)^2}{S(k+1)}. \tag{3.44}$$

Note that $S(k+1)$ can be lower-bounded as

$$\begin{aligned} S(k+1) &= \sum_{r=1}^{k+1} \frac{1}{\sqrt{r+1}} \geq \int_2^{k+3} \frac{1}{\sqrt{x}} dx \\ &= 2(\sqrt{k+3} - \sqrt{2}) \geq \nu\sqrt{k+3} \geq \nu\sqrt{k+1}, \end{aligned} \tag{3.45}$$

with $\nu = 2 - \sqrt{2}$, and where we employed monotonicity of $\frac{\sqrt{x+3}-\sqrt{2}}{\sqrt{x+1}}$ for $x \geq 1$. Moreover, we have that

$$\sum_{r=0}^{k} c(r)^2 = \sum_{r=0}^{k} \frac{1}{r+1} = \sum_{r=1}^{k+1} \frac{1}{r} \leq \int_1^{k+1} \frac{1}{x} dx + 1 \leq \ln(k+1) + 1. \tag{3.46}$$

The result of the Theorem 3, item *iii*), follows then from (3.41) by substituting (3.43)–(3.46), and setting $B_1 = \sum_{i=1}^{4} \frac{d_i}{\nu}$ and $B_2 = \frac{d_2}{\nu} + \frac{d_4}{\nu}$. Since (3.44) is valid for all $i = 1, \ldots, m$, we have that (via a direct application of triangle inequality)

$$\|\hat{x}_i(k) - \hat{x}_j(k)\|_2 \leq \sum_{i=1}^{m} \|\hat{x}_i(k) - \hat{v}(k)\| + \sum_{i=1}^{m} \|\hat{x}_j(k) - \hat{v}(k)\|,$$

which due to (3.45) and (3.46) then implies that the sequence $(\|\hat{x}_i(k) - \hat{x}_j(k)\|_2)_{k\in\mathbb{N}}$ converges to zero at a rate $\mathcal{O}(\frac{\ln k}{\sqrt{k}})$. This concludes the proof of item *i*).

Moreover, these relations also imply that the set of accumulation points of the sequence $(\hat{v}(k))_{k\in\mathbb{N}}$ coincides to that of the sequences $(\hat{x}_i(k))_{k\in\mathbb{N}}$, $i = 1, \ldots, m$. Hence, we conclude that all accumulation points of $(\hat{x}_i(k))_{k\in\mathbb{N}}$ are feasible due to the fact that all accumulation points of $(\hat{v}(k))_{k\in\mathbb{N}}$ are in $\cap_{i=1}^{m} X_i$ and the latter is a closed set, thus concluding the proof of item *ii*). This concludes the proof of Theorem 3.

Derivation of (3.43)

We first construct an upper-bound for the term on the left-hand side of (3.43). In fact, observe that

$$\begin{aligned} f(\hat{v}(k+1)) - f(x^\star) &= f\left(\frac{1}{S(k+1)}\sum_{r=1}^{k+1} c(r)\bar{v}(r)\right) - f(x^\star) \\ &\leq \sum_{r=1}^{k+1}\frac{c(r)}{S(k+1)} f(\bar{v}(r)) - f(x^\star) = \sum_{r=0}^{k}\frac{c(r+1)}{S(k+1)}\sum_{i=1}^{m}(f_i(\bar{v}(r+1)) - f_i(x^\star)) \\ &\leq \sum_{r=0}^{k}\frac{c(r)}{S(k+1)}\sum_{i=1}^{m}(f_i(\bar{v}(r+1)) - f_i(x^\star)), \end{aligned} \tag{3.47}$$

where the first equality follows by definition of $\hat{v}(k+1)$, the first inequality by convexity of f, the second equality by using the fact that $f = \sum_{i=1}^m f_i$ and changing the summation index, and the second inequality by using the fact that $c(r+1) = \frac{1}{\sqrt{r+1}} \leq \frac{1}{\sqrt{r}} = c(r)$ for all $r \in \mathbb{N}$.

In light of (3.42), for any $\beta_1 \in (0,1)$, a valid choice for the sequences $(\alpha_1(k))_{k\in\mathbb{N}}$ and $(\alpha_2(k))_{k\in\mathbb{N}}$ is $\alpha_1(k) = \alpha_2(k) = \alpha(k)$, where $\alpha(k) = a\Big(1 - \frac{1}{\sqrt{k+1}}\Big)$; to ensure that $1 - \beta_1 - \alpha_1(k) - \alpha_2(k) \geq 0$ as required by Lemma 4, item *ii*), it suffices to set $a = (1-\beta_1)/2$. Under these choices we have that

$$1 - \beta_1 - 2\alpha(k) = \frac{1-\beta_1}{\sqrt{k+1}} = (1-\beta_1)c(k). \tag{3.48}$$

Consider now (3.42) with the above choices for $\alpha_1(k)$ and $\alpha_2(k)$. Note that the series $\sum_{r=0}^{k}\sum_{i=1}^{m}\|x_i(r+1) - x^\star\|_2$ and $\sum_{r=0}^{k}\sum_{i=1}^{m}\|x_i(r) - x^\star\|_2$ are telescopic, thus all intermediate terms cancel. We now drop the terms involving $\|e_i(r+1)\|_2^2$ and $\|x_i(k+1) - x^\star\|_2$ as they are non-negative, and then divide the resulting expression by $2S(k+1) = 2\sum_{r=1}^{k+1}\frac{1}{\sqrt{r+1}}$ to obtain the following upper bound on the right-hand side of (3.47)

$$\begin{aligned} \sum_{r=0}^{k}\frac{c(r)}{S(k+1)}\sum_{i=1}^{m}(f_i(\bar{v}(r+1)) - f_i(x^\star)) &\leq \frac{\sum_{i=1}^m\|x_i(0) - x^\star\|_2^2}{2S(k+1)} + \frac{\beta_3}{2S(k+1)} \\ &+ \frac{\beta_2}{2}\sum_{r=0}^{k}\frac{c(r)^2}{S(k+1)} + mL^2\frac{1}{S(k+1)}\sum_{r=0}^{k}\frac{c(r)^2}{\alpha(r)}. \end{aligned} \tag{3.49}$$

By the right-hand side of (3.49), we obtain (3.43) with $d_1 = \frac{4mD^2+\beta_3}{2}$, $d_2 = \frac{\beta_2}{2} + \frac{4mL^2}{a}$. where, by Assumption 1, $\sum_{i=1}^m\|x_i(0) - x^\star\|_2^2 \leq 4mD^2$, with D defined as in Lemma 3, item i). Moreover, we used the fact that $\frac{c(r)^2}{\alpha(r)} = \frac{1}{a}\frac{\sqrt{r+1}}{\sqrt{r+1}-1}\frac{1}{r+1} \leq \frac{4}{a}c(r)^2$, due to monotonicity of $\frac{\sqrt{x+1}}{\sqrt{x+1}-1}$.

Derivation of (3.44)

Similarly to the derivation of (3.43), we apply the definition of both $\hat{x}_i(k)$, $i = 1,\dots,m$, and $\hat{v}(k)$ to upper-bound the left-hand side of (3.44) as

$$\begin{aligned} L\sum_{i=1}^m \|\hat{x}_i(k+1) - \hat{v}(k+1)\|_2 &= L\sum_{i=1}^m \left\| \frac{1}{S(k+1)} \sum_{r=1}^{k+1} c(r)\Big(x_i(r) - \bar{v}(r)\Big)\right\|_2 \\ &\leq \frac{L\mu}{S(k+1)} \sum_{r=1}^{k+1} c(r) \sum_{i=1}^m \|x_i(r) - v(r)\|_2, \end{aligned} \tag{3.50}$$

where the inequality follows from convexity of the norm. We will now construct an upper-bound on the right-hand side of (3.50). To this end, note that

$$\begin{aligned} \frac{L\mu}{S(k+1)} \sum_{r=1}^{k+1} c(r) \sum_{i=1}^m \|x_i(r) - v(r)\|_2 &= \frac{L\mu c(1)}{S(k+1)} \sum_{i=1}^m \|x_i(1) - v(1)\|_2 \\ &\quad + \frac{L\mu}{S(k+1)} \sum_{r=2}^{k+1} c(r) \sum_{i=1}^m \|x_i(r) - v(r)\|_2. \end{aligned} \tag{3.51}$$

We now invoke Lemma 3, item ii) – with r in the place of k, and t in the place of r – for the last term on the right-hand side of (3.51) so that

$$\begin{aligned} &\sum_{r=2}^{k+1} c(r) \sum_{i=1}^m \|x_i(r) - v(r)\|_2 = \sum_{r=1}^{k} c(r+1) \sum_{i=1}^m \|x_i(r+1) - v(r+1)\|_2 \\ &\leq 2\sum_{r=0}^{k} c(r) \sum_{i=1}^m \|e_i(r+1)\|_2 + m\lambda \sum_{i=1}^m \|x_i(0)\|_2 \sum_{r=0}^{k} c(r) q^r \\ &\qquad + m\lambda \sum_{r=1}^{k} c(r+1) \sum_{t=0}^{r-1} q^{r-t-1} \sum_{i=1}^m \|e_i(t+1)\|_2, \end{aligned} \tag{3.52}$$

where we added the term corresponding to $r = 0$ and used the fact that $c(r+1) \leq c(r)$ for all $r \in \mathbb{N}$, in first two terms on the right-hand side of (3.52). We analyse each term in the right-hand side of (3.52) separately. First, observe that

$$2\sum_{r=0}^{k} c(r) \sum_{i=1}^m \|e_i(r+1)\|_2 \leq \sum_{r=0}^{k} c(r)^2 + \sum_{i=1}^m \|e_i(r+1)\|_2^2, \tag{3.53}$$

using the identity $2xy \leq x^2 + y^2$. The intermediate term in the right-hand side of (3.52) can be manipulated to yield

$$m\lambda \sum_{i=1}^m \|x_i(0)\|_2 \sum_{r=0}^{k} c(r) q^r \leq \frac{m^2\lambda D}{1-q}, \tag{3.54}$$

since $c(r) \leq 1$ for all $r \in \mathbb{N} \cup \{0\}$, $\|x_i(0)\|_2 \leq D$ (Lemma 1) for all $i = 1, \ldots, m$, and using the closed-form expression for the sum of geometric series as $q \in (0,1)$. We deal with the last term in (3.52) in several steps. We start by expanding the terms to obtain

$$\begin{aligned}
&\sum_{r=1}^{k} c(r+1) \sum_{t=0}^{r-1} q^{r-t-1} \sum_{i=1}^{m} \|e_i(t+1)\|_2 \\
&= c(2) \sum_{i=1}^{m} \|e_i(1)\|_2 + c(3) \left(q \sum_{i=1}^{m} \|e_i(1)\|_2 \sum_{i=1}^{m} \|e_i(2)\|_2 \right) \\
&\qquad + \ldots + c(k+1) \left(\sum_{t=1}^{k} q^{k-t} \sum_{i=1}^{m} \|e_i(t)\|_2 \right). \qquad (3.55)
\end{aligned}$$

We now collect the terms containing the error vector $e_i(r)$, $r = 1, \ldots, k$, to obtain

$$\begin{aligned}
&m\lambda \sum_{r=1}^{k} c(r+1) \sum_{t=0}^{r-1} q^{r-t-1} \sum_{i=1}^{m} \|e_i(t+1)\|_2 = m\lambda \sum_{i=1}^{m} \|e_i(1)\|_2 \Big(c(2) + qc(3) + \ldots \\
&\qquad + q^{k-1} c(k+1) \Big) + \ldots + \sum_{i=1}^{m} \|e_i(k)\|_2 c(k+1) \\
&\leq \frac{m\lambda}{1-q} \sum_{r=1}^{k} c(r+1) \sum_{i=1}^{m} \|e_i(r)\|_2 \leq \frac{m\lambda}{1-q} \sum_{r=1}^{k} c(r) \sum_{i=1}^{m} \|e_i(r)\|_2 \\
&\leq \frac{m\lambda}{2(1-q)} \sum_{r=0}^{k} c(r)^2 + \frac{m\lambda}{2(1-q)} \sum_{r=0}^{k} \sum_{i=1}^{m} \|e_i(r+1)\|_2^2 \qquad (3.56)
\end{aligned}$$

where in the first inequality we used the fact that $q \leq \frac{1}{1-q}$ and $1 \leq \frac{1}{1-q}$ for any $q \in (0,1)$, while in the second inequality we used the fact that $c(r+1) \leq c(r)$. To obtain the last inequality we applied the relation $2xy \leq x^2 + y^2$ with $x = c(r)$ and $y = \|e_i(r+1)\|_2$, and then added the non-negative terms involving $c(0)^2$ and $\sum_{i=1}^{m} \|e_i(k+1)\|_2^2$. Substituting (3.51)–(3.54) and (3.56) into (3.50) we have that

$$\begin{aligned}
&L \sum_{i=1}^{m} \|\hat{x}_i(k+1) - \hat{v}(k+1)\|_2 \leq L\mu \left(1 + \frac{m\lambda}{2(1-q)} \right) \frac{\sum_{r=0}^{k} c(r)^2}{S(k+1)} \\
&+ \left(m\lambda + 2c(1) \right) \frac{L\mu m D}{S(k+1)} + \frac{L\mu}{S(k+1)} \left(1 + \frac{m\lambda}{2(1-q)} \right) \sum_{r=0}^{k} \sum_{i=1}^{m} \|e_i(r+1)\|_2^2. \qquad (3.57)
\end{aligned}$$

To obtain the result, we need to manipulate the last term in the right-hand side of (3.57). To this end, we invoke (3.42) with the same β_1 as in (3.48), but with $(\alpha_1(k))_{k \in \mathbb{N}}$ and $(\alpha_2(k))_{k \in \mathbb{N}}$ such that $\alpha_1(k) = \alpha_2(k) = \alpha$, for all $k \in \mathbb{N}$, following the same rationale as in Proposition 6 to obtain

$$\sum_{r=0}^{k}\sum_{i=1}^{m}\|e_i(r+1)\|_2^2 \leq \frac{\sum_{i=1}^{m}\|x(0)-x^\star\|_2^2+\beta_3}{1-\beta_1-2\alpha}+\frac{1}{1-\beta_1-2\alpha}\Big(mL^2\frac{2}{\alpha}+\beta_2\Big)\sum_{r=0}^{k}c(r)^2$$
$$\leq \frac{4mD^2+\beta_3}{1-\beta_1-2\alpha}+\frac{1}{1-\beta_1-2\alpha}\Big(mL^2\frac{2}{\alpha}+\beta_2\Big)\sum_{r=0}^{k}c(r)^2. \tag{3.58}$$

Substituting (3.58) into (3.57) we obtain (3.44) with constants

$$d_3 = L\mu\left[\left(1+\frac{m\lambda}{2(1-q)}\right)\frac{4mD^2+\beta_3}{1-\beta_1-2\alpha}+mD\Big(m\lambda+2c(1)\Big)\right],$$
$$d_4 = L\mu\left(1+\frac{m\lambda}{2(1-q)}\right)\left(1+\frac{1}{1-\beta_1-2\alpha}\Big(mL^2\frac{2}{\alpha}+\beta_2\Big)\right),$$

thus concluding the proof of Theorem 3.

4
Distributed actuator selection

In this chapter, we address a combinatorial optimisation problem, namely, the distributed actuator selection problem. As stated in Chapter 1, the main issue associated with such a problem is the presence of integer decision variables. We study a specific formulation of the problem that can be solved exactly by means of a convex relaxation and show how multi-agent optimisation schemes as the one studied in Chapter 3 can be employed to produce a solution to the corresponding relaxation.

4.1 Introduction

In the past few years, an active stream of research within the control community has been understanding and regulating complex networks with applications to health care, neuroscience, and social networks [76]. However, achieving an implementable solution in the aforementioned areas requires scalability and reliability.

One important problem arising from the study of large-scale complex networks is that of actuator and sensor selection/placement. The former aims to choose ν from m potential actuator positions to minimise some objective function, e.g., H_2 norm or controllability-related metrics, while the latter addresses a similar situation by deciding on ν sensors to minimise metrics related to estimation error. These problems have been extensively studied in the literature, ranging from applications in power systems [54] to the satellite assignment problem [119] and wireless networks [61].

Being combinatorial problems, actuator and sensor placement do not have, in general, efficient algorithms to determine their corresponding optimal solutions. The straightforward, albeit very commonly used, approach of enumerating all possible selections, and choosing the best one according to a given metric becomes intractable for high values of m. Even listing all possible alternatives requires shared information that in several applications might be considered as private.

An alternative approach is, instead of obtaining the optimal solution, to rely on algorithms that possess guaranteed sub-optimality bounds, for instance, the greedy algorithm applied to submodular functions [73], [89], [104]. Such algorithms have been successfully applied to the actuator and sensor selection problems [39], [65], [74], [139]. Furthermore, one can also sub-optimally solve the actuator and sensor placement problem using convex relaxations [45], [54], [68], [116], however, without *a priori* guaranteed sub-optimality bounds.

Within this context, this chapter aims to study the actuator selection problem using the trace of the controllability gramian as an optimisation metric. Under this metric and asymptotic stability of the dynamics, we show that the actuator placement problem can be equivalently posed as an Integer Linear Program (ILP). Using properties of integral polyhedra, we show through a sequence of reformulations that the optimal solution of this problem can be determined by means of a Linear Program (LP) without introducing any relaxation gap. This allows us to exploit recent results in the literature [48], and to determine the optimal solution by means of a primal-dual distributed algorithm, thus providing a scalable approach to the problem of actuator placement which has been up to now performed on a centralised manner enumerating all possible placement alternatives.

Deviating from recent attempts in the literature, we recognize the combinatorial nature of the problem but do not rely on submodularity properties of set functions, as in [65], [139], [140]. Our standpoint is closer to [68] since we use convex relaxations to study the problem. However, [68] focuses on different optimisation metrics and provides a sub-optimal solution to the problem. In contrast, this chapter adopts a particular, in some sense simpler, optimisation metric, but obtains stronger results, showing that the optimal solution of the actuator selection problem can be obtained by means of a linear program. The modularity of the metric used opens the road for a distributed algorithmic implementation which is of particular interest in large-scale complex networks.

The remainder of this chapter is organized as follows. Section 4.2 states the actuator selection problem under study, presents an equivalent ILP formulation, and provides an interpretation about the optimisation metric used in this chapter. In Section 4.3 we introduce some background notions based on the properties of integral polyhedra and show that the optimal solution of the ILP can be obtained by means of a LP. We also introduce two algorithms, one decentralised (based on dual decomposition) and one distributed, that generate optimal solutions for the problem under study. In Section 4.4 we illustrate the efficacy of our approach by means of a case study involving a simplified model of the European power grid. In Section4.5 we review the main results of this chapter.

4.2 The actuator selection problem

4.2.1 Problem Statement

We consider the actuator selection problem using the trace of the controllability gramian as the optimisation metric. To this end, let m denote the number of possible actuators and $S \subset \{1, \ldots, m\}$, with $|S| = \nu$, and consider

$$\dot{x}(t) = Ax(t) + B_S u_S(t), \tag{4.1}$$

where $A \in \mathbb{R}^{n \times n}$ is a matrix that represents the dynamics of the system and $B_S \in \mathbb{R}^{n \times \nu}$ the input matrix associated with the subset S. The state of this system is denoted by $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^\nu$ is the input. The objective is to choose a subset S of cardinality ν to maximise the trace of the controllability gramian, i.e.,

$$\underset{S \subset \{1,\ldots,m\},\ |S|=\nu}{\text{maximise}} \quad \mathbf{tr}(W_S), \tag{4.2}$$

where $AW_S + W_S A^\top + B_S B_S^\top = 0$.

We impose the following standing assumption on the matrix A that describes the system's dynamics.

Assumption 6. *Matrix A is Hurwitz, i.e., its eigenvalues have negative real part.*

Note that, similar to [68], we can reformulate problem (4.2) as the Boolean-convex problem

$$\begin{aligned} \underset{W,z}{\text{maximise}} \quad & \mathbf{tr}(W) \\ \text{subject to} \quad & AW + WA^\top + \sum_{i=1}^{m} z_i B_i B_i^\top = 0 \\ & \mathbf{1}^\top z = \nu, \quad z_i \in \{0,1\}, \ \forall\, i = 1,\dots,m, \end{aligned} \tag{4.3}$$

by associating decision variables z_i, $i = 1,\dots,m$, to each actuator, where [1] B_i, $i = 1,\dots,m$, is a column vector associated with each possible actuator. In the particular case where each actuator is connected to only one state we have $B_i = e_i$, for $i = 1,\dots,m$, where e_i is the standard unit vector. The symbol $\mathbf{1}$ stands for the vector of ones in $\mathbb{R}^m$.

Problem (4.3) can be simplified if we solve the Lyapunov equation. We can thus obtain the equivalent ILP formulation:

$$\begin{aligned} \underset{z\in\{0,1\}^m}{\text{minimise}} \quad & \sum_{i=1}^{m} c_i z_i \\ \text{subject to} \quad & \mathbf{1}^\top z = \nu \end{aligned} \tag{4.4}$$

where $c_i = -\mathbf{tr}(W_i), AW_i + W_i A^\top + B_i B_i^\top = 0$. Note that equivalence follows from uniqueness of the solution to the Lyapunov equation, which is guaranteed by Assumption 6, while c_i, $i = 1,\dots,m$, can be computed in parallel for each actuator position.

We could formulate problem (4.4) as one of minimising a modular function over a cardinality constraint in the set of feasible solutions, which would reduce the search for the optimal solution of (4.4) to a simple sorting problem. For

[1]For a total of m possible actuators, let B be a $n \times m$ matrix having B_i, $i = 1,\dots,m$, as its columns. In our notation, B_S represents a $n \times \nu$ matrix that has a B_i, $i = 1,\dots,m$, as one of its columns if and only if $i \in S$, i.e., B_S represents a selection of the columns of B according to a subset $S \subset \{1,\dots,m\}$ of cardinality ν.

$M \subset \mathbb{R}$, submodular function is a set function $f : 2^M \to \mathbb{R}$ with the property that $f(S \cup \{i\}) - f(S) \geq f(S \cup \{i,j\}) - f(S \cup \{j\}), \forall S \subset M \setminus \{i,j\}$. Intuitively, submodular functions have a diminishing return property, that is, the contribution of adding an element i deteriorates when the number of elements in S increases. Modular functions satisfy the previous inequality with equality. The study of combinatorial problems using modular and submodular functions has been the main subject in the recent literature [65], [73], [74], [139], especially because of guaranteed sub-optimal bounds of the greedy algorithm [89], [104].

Even though, in principle, the solution to (4.4) can be easily computed, this is not the case when m is large. Besides, privacy issues may also be a concern since some actuators may not be willing to share private data, e.g., the cost vector c_i. In this case, one could search for the optimal solution of the problem using distributed optimisation [11] as studied in Chapter 3.

4.2.2 Optimisation metric interpretation

We provide an interpretation of our choice for the optimisation metric. To this end, note that maximising the trace of the controllability gramian results in maximising the sum of its eigenvalues, which can be thought of as a proxy for average controllability [50], [139]. At the same time, considering full state access, $\mathbf{tr}(W_S)$ coincides with the H_2 norm of (4.1). Therefore, by solving (4.2) we are maximising the energy of the impulse response of all possible actuator placement alternatives under the hypothesis of full-state measurement. This conclusion complies with the physical interpretation presented in [81] (see also [112], [139] for further details).

4.3 Distributed implementation

4.3.1 ILP background

Let us consider the following ILP

$$\begin{aligned} \underset{z \in \mathbb{Z}_+^n}{\text{minimise}} \quad & \sum_{i=1}^{m} c_i^\top z_i \\ \text{subject to} \quad & \sum_{i=1}^{m} H_i z_i \leq g, \end{aligned} \tag{4.5}$$

where $z = [z_1^\top \dots z_m^\top]$, with $z_i \in \mathbb{Z}_+^{n_i}$, $\sum_{i=1}^m n_i = n$, is the decision variable, $g \in \mathbb{Z}^p$ is the resource vector, and $\sum_{i=1}^m H_i z_i \leq g$ is the coupling constraint, where $H_i \in \mathbb{Z}^{p \times n_i}$ – observe that $n_i = 1$ for all $i = 1, \dots, m$ in formulation (4.4), which implies $n = m$. Instances of problem (4.5) include, but are not limited to the knapsack and set covering problems [16]. Note that for the results of this subsection we allow z_i, $i = 1, \dots, m$, to be positive integer-valued but not necessarily binary as in (4.4).

A non-negative integer vector z is a feasible solution of (4.5) if it satisfies the coupling constraint. The set of all feasible solutions is called the feasible set, and if non-empty, (4.5) is said to be feasible. We define the set of optimal solutions as the subset of the feasible set such that the value of the objective function is less than or equal to the value of any other vector in the feasible set.

In general, solving an ILP problem is hard because of the difficulty in characterizing the convex hull of the feasible set in terms of polyhedral inequalities [16]. In this direction, [49], [149] provide upper bounds on the difference between the optimal solution of (4.5) and its convex relaxation by tightening the resource vector g. However, in some special cases, we can produce a convex relaxation that is exact, i.e., its optimal solution produces an optimal solution for (4.5). For instance, the celebrated Birkhoff-von Neumann theorem [10] states that the extreme points of the set of doubly stochastic matrices are permutation matrices. Using this theorem one can solve the assignment problem, where we have m objects to assign to m people and aim to find the allocation with minimum cost. Note that the optimal solution for this problem is a permutation matrix, and it is often known as allocation in *merit order*. The Birkhoff-von Neumann theorem provides a way to produce an optimal solution by minimising over the set of doubly stochastic matrices instead of using the integer formulation with permutation matrices.

Towards this direction, integral polyhedra possess important properties that allow solving an ILP up to optimality by means of its convex relaxation.

Definition 16. *A polyhedral set P in $\mathbb{R}^m$ is integral when all the extreme points of P have integer components.*

The intuition behind Definition 16 is that a linear function that is bounded over P attains its minimum at some vertex of P [10, Chapter 2, Prop. 2.4.2]. We are implicitly assuming that P has at least one vertex, which is the case if and only if P does not contain a line [10, Chapter 2, Prop. 2.1.2]. A related definition in this context is the notion of total unimodularity.

Definition 17. *Matrix $H \in \mathbb{Z}^{p\times m}$ is totally unimodular if the determinant of each submatrix is either* $0, 1,$ *or* -1.

The following result, whose proof is given in [16, Corollary 3.1], relates Definitions 16 and 17.

Lemma 6. *The polyhedron $P = \{x \in \mathbb{R}^m_+ | Hx \leq g, 0 \leq x \leq u\}$ is integral if and only if H is totally unimodular.*

Note that, similar to the Birkhoff-von Neumann theorem, Lemma 6 provides mechanisms to exactly solve an ILP through its convex relaxation. Indeed, as an immediate consequence of Lemma 6, we can argue that whenever matrix

$$H = \Big[H_1, \ldots, H_m\Big] \tag{4.6}$$

is totally unimodular the optimal solution of problem (4.5) is equal to the optimal solution of its convex relaxation (i.e., considering $z \in \mathbb{R}^m_+$).

4.3.2 Convex reformulation

Building on the results of Section 4.3.1, we now focus on the actuator selection formulation given by (4.4) and consider its convex relaxation

$$\begin{array}{ll} \underset{0\leq z_i\leq 1,\ \forall i=1,\ldots,m}{\text{minimise}} & \sum_{i=1}^{m} c_i z_i \\ \text{subject to} & \mathbf{1}^\top z = \nu \end{array} \tag{4.7}$$

Proposition 7. *The feasible set of* (4.4) *coincides with extreme points of the polyhedron*

$$P = \Big\{z \in \mathbb{R}^m | \mathbf{1}^\top z = \nu, \quad 0 \leq z_i \leq 1, \quad \forall i = 1, \ldots, m\Big\},$$

i.e., the set of feasible solutions of (4.7), *for all $\nu \in \mathbb{Z}_+$.*

Proof. First, note that P does not contain a line since $0 \leq z_i \leq 1$ for all $i = 1, \ldots, m$, hence it has at least one extreme point. Then, define H as in (4.6) with

$$H_i = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \forall i = 1, \ldots, m, \text{ and } g = \begin{bmatrix} \nu \\ -\nu \end{bmatrix}. \tag{4.8}$$

Now, let $\hat{H}$ be a square submatrix of H. We have two possible cases: $\hat{H}$ has dimension 1 or 2. In the former case, note that either $\hat{H} = 1$ or $\hat{H} = -1$, which trivially satisfies $\det(\hat{H}) = \pm 1$, where $\det(A)$ denotes the determinant of A. In the latter case, observe that every 2×2 submatrix has determinant zero. Therefore, by Definition 16, H is totally unimodular. The result follows directly from Lemma 6, thus concluding the proof. □

4.3.3 Primal-Dual algorithms

Proposition 7 shows that an optimal solution of (4.4) can be recovered by means of (4.7). However, the latter is a linear program. In this section we exploit this fact, and provide a decentralised and a distributed algorithm to obtain its optimal solution. To this end, we equivalently represent the constraints of (4.7) as $Hz \leq g$, where H and g are given by (4.8). The corresponding dual problem is given by

$$\underset{\lambda \geq 0}{\text{maximise}} \quad -\lambda^\top g + \sum_{i=1}^{m} \min_{0 \leq z_i \leq 1} (c_i + \lambda^\top H_i) z_i \tag{4.9}$$

The minimisation step required to evaluate the dual function can be performed in parallel for each $i = 1, \ldots, m$, making this formulation amenable to decomposition algorithms. Second, we have zero duality gap between problems (4.9) and (4.7) (by strong duality arguments in linear programming), which in turn provides an optimal solution for its integer formulation given in (4.4).

One way to solve problem (4.9) is to use dual ascent methods, which are encoded by

$$\begin{aligned} z_i(k+1) &\in \underset{0 \leq z \leq 1}{\text{argmin}} (c_i + (\lambda(k))^\top H_i) z, \quad i = 1, \ldots, m \\ \lambda(k+1) &\in \left[\lambda(k) + \alpha(k) \Big(\sum_{i=1}^{m} H_i z_i(k+1) - g \Big) \right]_+, \end{aligned} \tag{4.10}$$

where $[\cdot]_+$ represents the projection of its argument on the positive orthant. Typical choices for the time-varying step-size are $\alpha(k) = \beta/(k+1)$ or $\alpha(k) = \beta/\sqrt{k+1}$, for some $\beta > 0$. As shown in [9], with either choice for $\alpha(k)$, the sequence generated in (4.10) converges to the set of optimal solutions of (4.9).

Notwithstanding the progress made from the centralised problem (4.7) to the decentralised Algorithm (4.10), this approach still has some drawbacks. At each iteration, the dual variable λ needs to be broadcast in order to perform the next primal update, which must be sent to the central processor that performs the dual update.

To alleviate the centralised dual update step we introduce a time-varying communication network with edge weights $a_{ij}(k)$, where $a_{ij}(k) = 0$ implies that node j does not share information with node i at iteration k. Similarly as in Chapter 3, we impose the following assumptions on the matrix $A(k)$, whose entries are given by the elements $a_{ij}(k), i, j \in \{1, \ldots, m\}$.

Assumption 7. *We assume that:*

i) The graph $(\mathcal{N}, \mathcal{E}_\infty)$ is strongly connected. Moreover, there exits a uniform upper bound on the communication time for all $(j, i) \in \mathcal{E}_\infty$.

ii) There exists $\tau \in (0, 1)$ such that for all $k \in \mathbb{N}$ and for all $i, j = 1, \ldots, m$, $a_{ii}(k) \geq \tau$, and if $a_{ji}(k) > 0$ then we have that $a_{ji}(k) \geq \tau$.

iii) Matrix $A(k)$ is doubly stochastic.

We are now in a position to present Algorithm 2, which was proposed in [48], and in contrast to (4.10) offers a distributed implementation. We will henceforth refer to each node i, $i = 1, \ldots, m$, as agent, which interacts with other agents according to the aforementioned communication protocol. By Algorithm 2, each agent calculates a weighted sum of the dual variables that were calculated by neighbouring agents according to the underlying communication structure at the previous iteration (Step 6) to update its own dual variable (Step 8), eliminating the need for a central

Algorithm 2 Distributed algorithm of [48]

Require: : $g, H_i, c_i, \quad \forall i = 1, \ldots, m$
1: $z_i^0 \in [0, 1], \quad \forall i = 1, \ldots, m$
2: $\lambda_i^0 = 0, \quad \forall i = 1, \ldots, m$
3: $k = 0$
4: **while** convergence is not achieved **do**
5:    **for** $i = 1$ to m **do**
6:       $\ell_i(k) = \sum_{j=1}^{m} a_{ij}(k)\lambda_j(k)$
7:       $z_i(k+1) \in \text{argmin}_{0 \le z \le 1}(c_i + (\ell_i(k))^\top H_i)z$
8:       $\lambda_i(k+1) \in [\ell_i(k) + \alpha(k)\Big(H_i z_i(k+1) - \frac{g}{m}\Big)]_+$
9:       $\hat{z}(k+1)_i = \hat{z}_i(k) + \frac{\alpha(k)}{\sum_{r=1}^{k} \alpha(k)}(z_i(k) - \hat{z}_i(k))$
10:    **end for**
11:    $k \leftarrow k + 1$
12: **end while**

agent to perform the dual update. Additionally, observe that variable $\alpha(k)$ is a time-varying step size, with the following property.

Assumption 8 (Time-varying step size)**.** *The sequence* $(\alpha(k))_{k \ge 0}$ *is non-increasing, positive,* $\sum_{k=1}^{\infty} \alpha(k) = \infty$*, and* $\sum_{k=1}^{\infty} (\alpha(k))^2 < \infty$*.*

A common choice for the time-varying step is $\alpha(k) = \frac{\beta}{k+1}$, for some $\beta > 0$. Assumptions 7 and 8 are essential for the convergence proof (see [48] for details).

Note that in both Algorithms 2 and (4.10) the primal update is performed by the following "if-else" clause

$$z_i(\zeta) = \underset{0 \le z \le 1}{\text{argmin}}(c_i + \zeta^\top H_i)z = \begin{cases} 1, & c_i + \zeta^\top H_i \le 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\zeta = \lambda(k)$ in (4.10) and $\zeta = \ell_i(k)$ in Algorithm 2. As a consequence, Step 7 in Algorithm 2 and the primal update in (4.10) is computationally inexpensive.

Under assumptions 7 and 8, it is shown in [48] that $\lim_{k \to \infty} \text{dist}(\hat{z}(k), Z^\star)$ is equal to zero, where $\hat{z}(k)$ is the sequence generated by Algorithm 2, $Z^\star$ is the set of optimal solution of (4.7) and $\text{dist}(x, X)$ is the distance between the point x and the set X.

Theorem 4. *Consider the actuator selection formulation given by* (4.2) *and suppose Assumptions 7 and 8 hold. Then the sequence* $\hat{z}(k)$ *generated by Algorithm 2 produces the optimal objective value for* (4.2)*.*

Proof. By the proof of Theorem 2 in [48] the sequence $(\hat{z}(k))_{k\geq 0}$ converges to the set of optimal solutions of (4.7), which, through Proposition 7, implies that the optimal objective function values of (4.7) and (4.4) coincide. However, by the equivalence between (4.4) and (4.2) (see Section 4.2) the result follows. □

The proof of Theorem 4 relies on several equivalences. These can be summarised as

$$(4.2) \overset{\star}{\iff} (4.4) \overset{Prop. 7}{\iff} (4.7) \overset{\#}{\iff} (4.9).$$

The left-hand side equivalence ($\star$) stands for the formulation of the actuator selection problem as an ILP through the steps presented in Section 4.2. Some ideas used in this process appear in [68], [140], but the last step where we convert into an ILP had not been explored in the literature. The right-hand side equivalence (#) is due to linear programming strong duality (see [11], [48]). The intermediate, instrumental equivalence result, was established by means of Proposition 7.

Under uniqueness[2] of the solution of (4.7), we can strengthen the result of Theorem 4.

Corollary 1. *Assume* (4.7) *admits a unique solution and let* $\xi = \lim_{k\to\infty} \hat{z}(k)$. *We then have that the optimal solution* $S^\star$ *of* (4.2) *is given by* $S^\star = \{i | \xi_i = 1\}$.

Proof. According to Theorem 4, the optimal values of (4.7) and (4.2) are the same. Since the solution of (4.7) is unique, we know that $\xi = \lim_{k\to\infty} \hat{z}(k)$ is well-defined and, in addition, it is an extreme point of the set of feasible solutions of (4.7) (by Proposition 7). Hence, it is a feasible solution for (4.2). Due to Theorem 4 it is also optimal since ξ will achieve the optimal value of (4.7). By the equivalence of (4.7) and (4.2), $S^\star$ constitutes the optimal solution of (4.2), thus concluding the proof. □

Remark 2. *One can prove that a similar result holds for the decentralised algorithm in* (4.10), *namely, at the limit, the sequence* $z(k)$ *generated by* (4.10) *converges in*

[2] Even without uniqueness of the solution of (4.7) we can always produce an optimal solution for (4.2), however, this solution would differ according to the iteration index (corresponding to practical convergence) where Algorithm 1 or iteration (9) is terminated.

terms of value to the optimal value of (4.2)*. For the sake of brevity, we do not present the proof here, however, it follows from the proof of Theorem 4 by setting the weights in Assumption 7 to be iteration invariant, and all of them equal to* $1/m$*. Moreover, it is worth pointing out that our main result does not depend on the particular algorithmic choice, and other distributed or decentralised schemes could be employed as well.*

4.4 Numerical example: European power grid

In this section, we illustrate the proposed method for the actuator selection problem using a case study involving a simplified model of the European power grid to decide the location of HVDC links in the network.

In particular, we revisit the HVDC allocation problem studied in [139]. In general, HVDC links are employed to enhance transient response of the power system by influencing active and reactive power injections to damp frequency oscillations and prevent rotor angle instability [53], [139]. The model consists of a linear system that represents the European grid, which is composed of 74 buses connected to a generator and a constant impedance load. The linear model is obtained after linearising the swing equations about nominal operating points for each possible HVDC link placement. As in [139], the purpose of this example is to assess the efficacy of the proposed algorithm in a realistic setting, while from an application point of view further investigation is needed, as in HVDC placement decisions in the European power system controllability is just one of several objectives, e.g., economic. More details about the model can be found in [53], [54] and references therein.

The linearised model has 148 states, since each bus consists of one generator and each generator has two state variables corresponding to rotor angle and frequency dynamics. Following [139] we suppose that any generator can be possibly connected to any other generator in the grid, which yields 2701 possible connections, from which we want to select the best 10 placements according to the controllability trace optimisation metric. Simple calculations show that this configuration gives us a total of approximately 5.6×10^{27} possibilities, which is far beyond a sorting algorithm enumerating all alternatives can handle.

To implement Algorithm 2 all agents receive matrix A, and each agent its own matrix B_i, $i = 1, \ldots, 2701$, from the network. Upon this, agents can locally compute their own c_i by solving the corresponding Lyapunov equation.

The convergence rate of Algorithm 2 can be improved by introducing a sequence $\tilde{z}_i(k)$ defined as

$$\tilde{z}_i(k+1) = \begin{cases} \hat{z}_i(k+1), & k < L, \\ \frac{\sum_{r=L}^{k} \alpha(r) z_i(r+1)}{\sum_{r=L}^{k} \alpha(r)}, & k \geq L, \end{cases}$$

where L is defined to be a fraction of the total number of iterations. Similar arguments used to prove convergence of sequence $(\hat{z}(k))_{k\geq 0}$ of Algorithm 2 to the optimal solution of (4.7) applies to $(\tilde{z}(k))_{k\geq 0}$ [48, Theorem 2]. The latter sequence alleviates the influence of bad estimates of the dual variable in earlier steps of Algorithm 2.

In our simulations, we initialise the primal and dual variables to zero, as suggested in steps 1 and 2 of Algorithm 2. Furthermore, we define the iteration-varying step size as $\alpha(k) = \beta/(k+1)$, with $\beta = 10$, and run 2000 iterations. We also create a time-varying communication structure to satisfy Assumption 7 by alternating between two strongly connected graphs, i.e., $a_{ij}(2r)$ and $a_{ij}(2r+1)$ are constant for all $r \in \mathbb{Z}_+$. To guarantee the other conditions of Assumption 7 for each graph, we generate a doubly stochastic matrix by forming a convex combination of 100 randomly generated permutation matrices, making sure we include the identity matrix to ensure $a_{ii}(k) \geq \eta$ with $\eta \in (0,1)$. If the generated matrix satisfies $a_{ij}(k) \geq \eta$ whenever $a_{ij}(k) > 0$ we return the matrix; in the negative case, we repeat the process until these assumptions are achieved. Besides, if this algorithm terminates, Assumption 7 is satisfied with $T = 2$.

Figure 4.1 shows the evolution of the objective function (top graph) and the constraint violation for the sequences $\hat{z}(k)$ (solid blue line) and $\tilde{z}(k)$ (dashed red line). In the top graph, the optimal value of the centralised problem counterpart is shown by means of a dashed green line. Note that, at the beginning, the algorithm exhibits superior performance as far as the optimal value is concerned, however, the generated primal iterates are infeasible. As the algorithm progresses, we degrade the value of the objective to achieve primal feasibility. Additionally, as we can see, the sequence $\tilde{z}(k)$ has a better convergence rate than sequence $\hat{z}(k)$ (we use $L = 1000$ in our simulation).

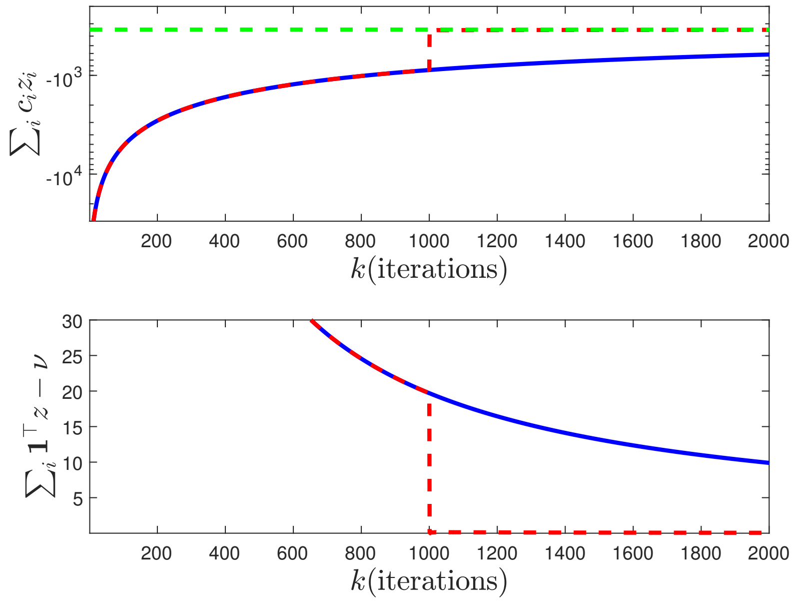


Figure 4.1: Evolution of the objective function (top graph) and constraint violation (bottom graph) for the sequences $\hat{z}(k)$ (solid blue line) and $\tilde{z}(k)$ (dashed red line) of Algorithm 2 for the European power model with $\alpha(k) = \frac{10}{(k+1)}$. In the top graph, the dashed green line corresponds to the solution of the centralised problem counterpart.

For completeness, we also apply the decentralised algorithm whose main steps are encoded by (9). As a stopping criterion for Algorithm 2 we use primal feasibility. Figure 4.2 illustrates the results with the choice of $\alpha(k) = 0.1/\sqrt{(k+1)}$ for the time-vanishing step size. Observe that the optimal solution is achieved when we find a primal feasible solution, which occurs around 3640 iterations (see zoomed areas).

Both Algorithm 2 and (4.10) converge to the optimal solution of the problem, which in this case admits a unique solution (see Corollary 1), the former being more adequate for large scale networks because it does not need the dual variable to be updated by a central processor. It should be mentioned that the parameter β

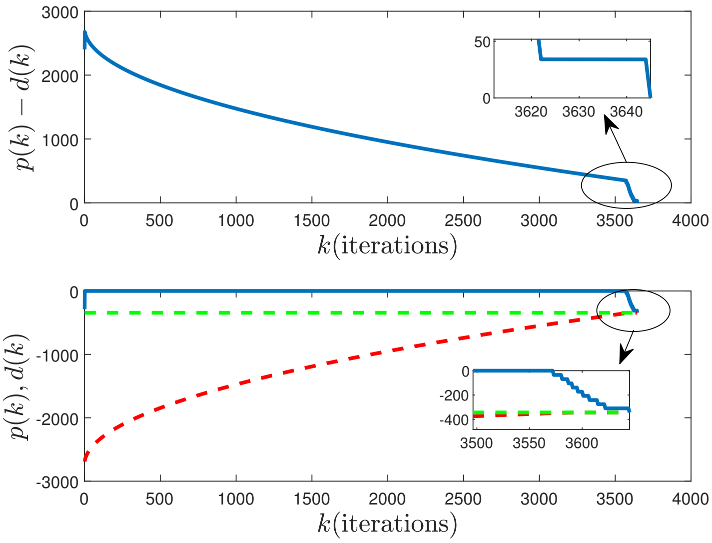


Figure 4.2: Duality gap (top) and values of the primal and dual objective values (bottom) for the dual algorithm given in (4.10). In the bottom graph, the primal objective value at each iteration is denoted by the solid blue line; the dual objective by the dashed red line; and the centralised optimal by the dashed green line. A primal feasible solution is found after approximately 3640 iterations.

of the time-varying step-size, albeit not interfering in the theoretical convergence, is decisive for the rate of convergence.

4.5 Conclusion

In this chapter we have addressed the second challenge of current optimisation methods stated in Chapter 1, namely, the presence of integer decision variables. The main results of this chapter used properties of integral polyhedra to solve a specific formulation of the actuator selection problem. Indeed, if the trace of the controllability gramian is employed as the optimisation metric we have shown that the optimal solution of the actuator selection problem can be obtained by means of a linear program. To solve the resulting LP, we provided two primal-dual algorithms: a decentralised one based on dual decomposition; and a distributed one that does not require a common processor for dual updates. The efficacy of our methods was investigated numerically on a European power grid case study.

Up to this point, we have studied two of the main challenges exposed in Chapter 1. In the next two chapters of the thesis we focus on the third one by addressing uncertain optimisation problems using the scenario approach theory.

5 Scenario based optimisation: Trading feasibility to performance

This chapter deals with optimisation in the presence of uncertain constraints, which constitutes one of the challenges mentioned in Chapter 1. Specifically, we study a randomised approximation to chance-constrained optimisation under the lens of the scenario approach theory.

5.1 Introduction

Uncertain optimisation programs capture a wide class of engineering applications. Tractability of this class of optimisation problems is an active area of research [2], [23], [32], [34], [35], [93], [121], [146]. In the last decades, several approaches have been developed to cope with uncertainty in an optimisation context. Among those, robust optimisation [6], [7], [14], [15] has been successfully applied to several control problems [26], [55], [60], [107], [126], [141]. It consists of making certain assumptions, often arbitrary, on the geometry of the uncertainty set (ellipsoidal, polytopic, etc.) and then optimizing over the worst-case performance within this set. Another approach is chance-constrained optimisation [105], [111], [118] (see also Section 2.3.3) that relies on imposing constraints that only need to be satisfied with given probability. However, these problems are hard to solve in general, without imposing any assumption on the underlying distribution of the uncertainty (e.g., Gaussian).

Within this context, this chapter lies in the realm of the scenario approach theory [22], [24], [25], [29]–[32], [56], [94]: a randomised technique which involves generating a finite number of scenarios and enforcing a different constraint for each of them, as introduced in Section 5.2 of Chapter 2. Under convexity, the optimal solution to such a scenario program is shown to be feasible (with certain probability) to the associated chance-constrained program.

This chapter capitalises on the fact that the bound of the sampling-and-discarding scheme is not tight. We provide a novel analysis for a specific removal scheme that involves solving a cascade of scenario programs and removing, at each stage, a number of scenarios equal to the number of decision variables in each scenario program. Our analysis culminates in a tighter bound than the one in [30]. This bears important consequences in the application of the scenario theory to some control problems [25], [27], [33], [42], [44], [66], [75], [90], [131], [135], [136], [145], as we may be able to achieve better performance while guaranteeing the same level of constraint violation and confidence. The proposed bound on the probability of constraint violation is similar to [22], [30] in that it is distribution-free and holds, under a non-degeneracy assumption, for all convex problems. We also characterise a class of scenario programs for which the proposed bound holds with equality, thus showing its tightness.

To summarise, our main contributions are: (1) proposing and analysing a removal scheme that possesses tighter guarantees than [22], [30] on the probability of constraint violation associated with the solution of scenario programs with discarded constraints; (2) proving tightness of the proposed bound by characterising the class of scenario programs that satisfies our bound with equality; (3) relaxing an assumption present in [30] that requires the removed scenarios to be violated by the resulting solution; and (4) developing a novel proof line. Our analysis departs from the one of [30], and is based on probably approximately correct (PAC) learning concepts that use the notion of compression [51], [94], [147].

It is important to highlight that our analysis holds for a particular discarding scheme, which requires removing scenarios in batches. Extension to this direction is outside the scope of the current chapter. Moreover, all our results are *a priori*; possibly less conservative but *a posteriori* results are available [32], [34], [56], however, follow a different conceptual and analysis line from the one adopted in this chapter.

The chapter is organized as follows: Section 5.2 reviews the some results of the scenario approach theory and recalls the main theorem on compression learning theory that was presented in Section 2.4, as it will serve the basis for our developments. Section 5.3 introduces the proposed scenario discarding scheme and states the main results of the chapter, while proofs are provided in Section 5.4. Section 5.5 provides a class of optimisation programs for which the proposed result is tight, while Section 5.6 illustrates the theoretical results by means of a numerical example. A summary of the results of this chapter and a connection with the main theme of this thesis is presented in Section 5.7.

5.2 Scenario approach theory

This section reviews the main results of the scenario approach theory, showing how they can be used to produce, with high-probability, a feasible solution to chance-constrained optimisation problems (see Section 2.3.3 for more details). The scenario approach theory assumes that m i.i.d. samples $S = \{\delta_1, \dots, \delta_m\}$ are available and studies the properties of the optimal solution of problem

$$\begin{aligned} \underset{x \in \mathcal{X}}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta_i) \leq 0, \text{ for all } \delta_i \in S \setminus R(S), \end{aligned} \tag{5.1}$$

where $\mathcal{X}$ is a convex, compact set and g is a measurable function in the second argument as in Section 2.3.3, and $g(\cdot, \delta) : \mathbb{R}^d \to \mathbb{R}$ is convex for all $\delta \in \Delta$, $S = \{\delta_1, \dots, \delta_m\}$ is a set of i.i.d. samples from $\mathbb{P}$, also called scenarios, and $R : S \to 2^S$ is a mapping that takes as input S and returns a subset of S containing scenarios that are discarded by means of some removal procedure. Throughout this chaper the cardinality of $R(S)$ is denoted by r. In the literature [24], [29], [31], problem

(5.1) is referred to as scenario program since one constraint is enforced for each scenario in $S \setminus R(S)$. Throughout this thesis, we impose the following assumption.

Assumption 9. *The scenario program* (5.1) *admits a unique solution for any* $S = \{\delta_1, \ldots, \delta_m\}$, $m \geq d$. *Moreover, its feasible set has a non-empty interior.*

Assumption 9 is standard within the scenario approach theory [22], [29], [30], [32]. Denote by $x^\star(S)$ the (unique) optimal solution of (2.20), where the dependence on the scenarios in S is made explicit. Some concepts at the core of the scenario approach theory are presented in the sequel [24], [29].

Definition 18 (Support constraint)**.** *Consider the scenario program* (5.1)*. A scenario in* $S \setminus R(S)$ *is said to be a support constraint (or support scenario) if its removal changes the optimal solution of* (5.1)*. The set of all support constraints is called the support set of* (5.1)*, and will be denoted by* $\mathrm{supp}(x^\star(S))$*.*

Definition 19 (Fully-supported problems)**.** *A scenario program* (5.1) *is said to be fully-supported if for all* $m \in \mathbb{N}$ *the cardinality of the support set is equal to* d *with probability one with respect to* $\mathbb{P}^m$*.*

Definition 20 (Non-degenerate programs)**.** *A scenario program* (5.1) *is non-degenerate if, with* $\mathbb{P}^m$*-probability one, solving the problem by enforcing the constraints only on the support set,* $\mathrm{supp}(x^\star(S))$*, results in* $x^\star(S)$*, i.e., the solution obtained when all samples in* $S \setminus R(S)$ *are employed.*

Note that if a problem is fully-supported then it is also non-degenerate, however, the opposite implication does not hold. Moreover, in a convex optimization context, non-degeneracy is a relatively mild assumption, and implies that scenarios give rise to constraints at general positions that lead to scenario programs with non-empty support sets. On the contrary, requiring a scenario program to be fully-supported is much stronger. However, it exhibits interesting theoretical properties as the number of support scenarios is exactly equal to the dimension of the decision vector d [29], [94].

If scenarios are not removed (i.e., $R(S) = \emptyset$) in (5.1), the authors in [29] have shown an upper bound on the probability of constraint violation for the optimal solution of (5.1), which is presented in the sequel.

Theorem 5 (Theorem 1, [29]). *Consider Assumption 9 and fix $\epsilon \in (0, 1)$. Let $m \in \mathbb{N}$ be given and denote by $x^\star(S)$ the optimal solution of (5.1) with $R(S) = \emptyset$. Then we have that*

$$\mathbb{P}^m \left\{ S \in \Delta^m : \mathbb{P}\{\delta : g(x^\star(S), \delta) > 0\} > \epsilon \right\} \leq \sum_{i=0}^{d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}. \tag{5.2}$$

The left-hand side of Theorem 5 is composed of two nested probabilities: the outer one represents the confidence with which the bound is valid, and the inner one stands for the risk incurred by the optimal solution of problem (5.1). An important feature of the bound in (5.2) is the fact that it holds for all convex optimization problems and all distributions $\mathbb{P}$. In fact, [29] showed that the class of fully-supported optimization problems achieves the bound of Theorem 5. In this sense, the bound in (5.2) is said to be tight.

On the other hand, if $R(S)$ is not empty, the authors in [22], [30] have studied the probability of constraint violation associated to the optimal solution of (5.1), establishing the following result.

Theorem 6 (Theorem 2.1, [30]). *Consider Assumption 9, and fix $\epsilon \in (0, 1)$. Let $m > d+r$ and denote by $x^\star(S)$ the optimal solution of (5.1). If all removed scenarios are violated by $x^\star(S)$, i.e., $g(x^\star(S), \delta) > 0$ for all $\delta \in R(S)$, with $\mathbb{P}^m$-probability one, then*

$$\mathbb{P}^m \left\{ S \in \Delta^m : \mathbb{P}\Big\{ \delta \in \Delta : g(x^\star(S), \delta) > 0 \Big\} > \epsilon \right\} \leq \binom{r+d-1}{r} \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}. \tag{5.3}$$

Theorem 6 is known in the literature as the sampling-and-discarding approach to scenario programs [30], or as scenario approach with discarded constraints [22]. Note that by allowing scenarios to be removed (i.e., $R(S) \neq \emptyset$), the sampling-and-discarding approach enables the decision maker to improve the optimal objective value with respect to the case when scenarios are not discarded, while keeping the probability of constraint violation under control.

However, as opposed to Theorem 5, the bound of Theorem 6 is not tight. Indeed, in Section 4.2 of [30], the authors show that there exists a class of convex optimization programs and a discarding scheme such that the right-hand side of (5.3) can be replaced by $\sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}$. This argument is not constructive and is limited to an existential statement.

Motivated by the fact that Theorem 6 is not tight, in the next section we focus on a specific removal scheme composed by a cascade of scenario programs and employ Theorem 1 to perform the analysis of the resulting probability of constraint violation associated to (5.1). Recall that Theorem 1 builds on a compression learning theoretic framework and establishes that if a unique compression set exists (see Definition 15, Chapter 2), then

$$\mathbb{P}^m\{S \in \Delta^m : d_{\mathbb{P}}(\mathcal{A}(S), T) > \epsilon\} = \sum_{i=0}^{\zeta-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}, \tag{5.4}$$

where $\mathcal{A} : \Delta^m \to 2^\Delta$ is a mapping that takes i.i.d. samples $S = \{\delta_1, \dots, \delta_m\}$ from the unknown probability distribution $\mathbb{P}$ as input and outputs a subset of Δ.

Let $T = \Delta$ and fix $\epsilon \in (0, 1)$. Observe that the right-hand side of (2.16) goes to zero as m tends to infinity. This is a desirable property, as it indicates that Δ can be asymptotically approximated by $\mathcal{A}(C)$. Moreover, for a fixed $m \in \mathbb{N}$, the result of Theorem 1 provides a non-asymptotic result, quantifying the measure of the set $\Delta \setminus \mathcal{A}(C)$. A mapping with this property is called PAC within the learning literature. Hence, we can reinterpret the result of Theorem 1 as stating that if a mapping possesses a unique compression set, then it is at least $(1-\epsilon)$-accurate as an approximation of Δ (approximately correct), with confidence (probably) equal to $1 - \sum_{i=0}^{\zeta-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}$.

5.3 Proposed discarding scheme and main results

In this section, we present a particular scenario discarding scheme which does not require that all removed scenarios are violated by the resulting solution. For a given set of scenarios $S = \{\delta_1, \ldots, \delta_m\}$, we solve a cascade of $\ell + 1$, where $(\ell + 1)d < m$, optimisation programs denoted by $P_k,\ k \in \{0, \ldots, \ell\}$. For each $k \in \{1, \ldots, \ell\}$, the corresponding program is given by

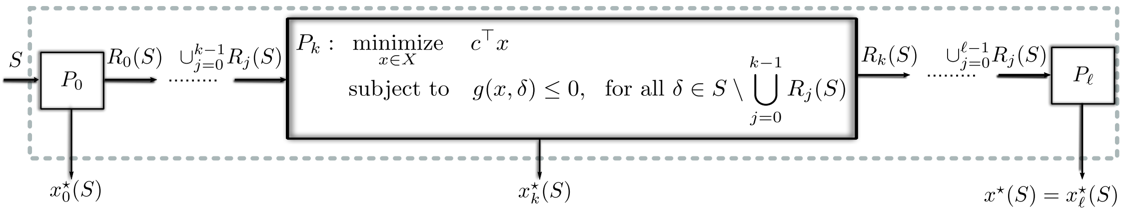


Figure 5.1: Block diagram of the proposed scheme. For a given set of scenarios $S = \{\delta_1, \ldots, \delta_m\}$ with $(\ell + 1)d < m$, we solve a cascade of $\ell + 1$ optimisation programs denoted by $P_k,\ k \in \{0, \ldots, \ell\}$. For each k, $k \in \{0, \ldots, \ell - 1\}$, we remove $R_k(S)$ scenarios with $|R_k(S)| = d$, hence, in total $r = \ell d$ scenarios (the ones in $\bigcup_{j=0}^{\ell-1} R_j(S)$) are discarded. The choice of each set of discarded scenarios depends on the initial set S, thus we introduce it as argument of R_k. If each problem is fully-supported, the elements of $R_k(S)$ correspond to the (unique) set of support scenarios associated with the minimizer $x_k^\star(S)$ of that program – see (5.5); otherwise, $R_k(S)$ contains the support scenarios as well as additional scenarios selected according to a lexicographic order regularization procedure, as in (5.8). The final solution is denoted by $x^\star(S) = x_\ell^\star(S)$.

$$
\begin{aligned}
P_k :\ & \underset{x \in X}{\text{minimise}} \quad c^\top x \\
& \text{subject to} \quad g(x, \delta) \le 0, \quad \text{for all } \delta \in S \setminus \bigcup_{j=0}^{k-1} R_j(S),
\end{aligned}
$$

where $R_k(S)$, with $|R_k(S)| = d$, represents the set of removed scenarios at stage k, and $\bigcup_{j=0}^{k-1} R_j(S)$ the ones that have been removed up to stage k. For $k = 0$, we solve problem P_0 by enforcing all the scenarios in S. Notice that the number of removed scenarios at stage ℓ is given by ℓd (the samples in the set $\bigcup_{j=0}^{\ell-1} R_j(S)$). The choice of each set of discarded scenarios depends on the initial set S, thus we introduce it as an argument in R_k. A schematic illustration of the proposed scheme is provided in Figure 5.1.

Our choice for $R_k(S)$, $k \in \{0, \dots, \ell - 1\}$, will be detailed in the following two subsections, and it relies on the properties of each minimizer $x_k^\star(S)$. We distinguish two cases according to whether the underlying problem is fully-supported or only non-degenerate (both of these concepts have been defined in Section 5.2, Chapter 2).

5.3.1 The fully-supported case

Throughout this section the cardinality of the support set of problem P_k, $k = 0, \dots, \ell$, is assumed to be equal to d, which is the dimension of the optimisation variable, with $\mathbb{P}^m$-probability one. We formalize this statement in the following assumption.

Assumption 10 (Fully-supportedness)**.** *For all $k \in \{0, \dots, \ell\}$, P_k is fully-supported with $\mathbb{P}^m$-probability one.*

Under Assumption 10, our choice for the removed scenarios is given by

$$R_k(S) = \mathrm{supp}(x_k^\star(S)), \quad k \in \{0, \dots, \ell - 1\}, \tag{5.5}$$

i.e., we remove the support set of the corresponding optimal solution of P_k. Note that the cardinality of $R_k(S)$ is equal to d and this choice for the removed scenarios guarantees that the objective function decreases at each stage, thus improving performance. Moreover, for $k = \ell$, we denote by $R_\ell(S)$ the support set of $x_\ell^\star(S)$; this quantity will be used in the sequel. Note that $R_\ell(S)$ does not contain any removed scenarios.

One of the main results of this chapter is to tighten the bound of Theorem 6 under Assumption 10. This is achieved in the following theorem.

Theorem 7. *Consider Assumptions 9 and 10. Fix $\epsilon \in (0, 1)$, set $r = \ell d$ and let $m > r + d$. Consider also the scenario discarding scheme as encoded by* (5.5) *and illustrated in Figure 5.1, and let the minimizer of the ℓ-th program be $x^\star(S) = x_\ell^\star(S)$. We then have that*

$$\mathbb{P}^m\Big\{S \in \Delta^m : \mathbb{P}\Big\{\delta \in \Delta : g(x^\star(S), \delta) > 0\Big\} > \epsilon\Big\} \leq \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1 - \epsilon)^{m-i}. \tag{5.6}$$

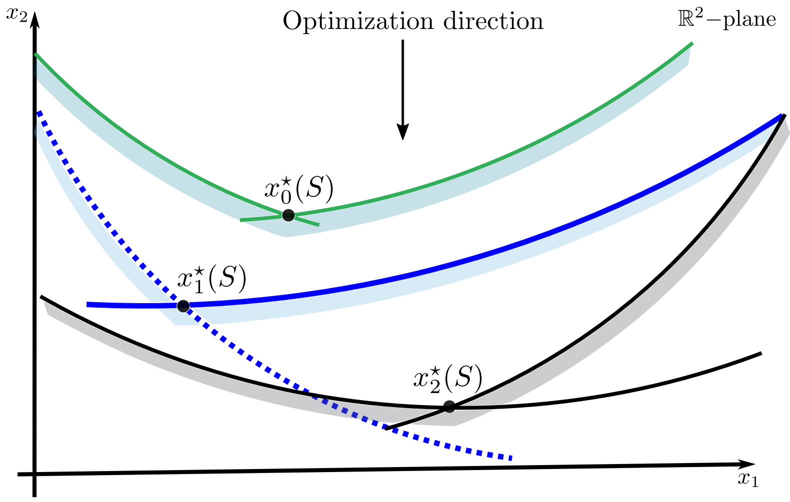


Figure 5.2: Pictorial example that illustrates the scheme proposed in Figure 5.1 for fully-supported problems. In this case, we have that $d = 2, m = 6, r = 4$, and $\ell = 2$, and all the problems P_k, $k \in \{0, 1, 2\}$, satisfy Assumption 10. The objective function is given by $c^\top x = x_2$ (indicated by the downwards pointing arrow). The constraint sets are denoted by the different colors patterns: the green constraints are associated to the samples of $\text{supp}(x_0^\star(S))$, the blue ones to $\text{supp}(x_1^\star(S))$, and the black ones correspond to $\text{supp}(x_2^\star(S))$. Observe that the dashed-blue constraint is removed by the scheme of Figure 5.1, but it is not violated by $x_2^\star(S)$.

The proof of Theorem 7 is deferred to Section 5.4.1. Note that unlike [22], [30], we do not require the removed scenarios to be violated by the resulting solution (see Figure 5.2).

To illustrate how the proposed scheme works, we consider the pictorial example of Figure 5.2. Note that $d = 2$, $m = 6$, and we remove $r = 4$, thus requiring $\ell = 2$ steps of the removal scheme of Figure 5.1. All the problems P_k, $k \in \{0, 1, 2\}$, are fully-supported, thus satisfying Assumption 10. The objective function is given by $c^\top x = x_2$ and is indicated by the downwards pointing arrow. The corresponding solution for the intermediate problems is illustrated by $x_k^\star(S)$, for $k \in \{0, 1, 2\}$, and the support set of each stage by different colour patterns. For instance, the green constraints are the support set, namely, $\text{supp}(x_0^\star(S))$, of problem P_0. The shaded colour under each constraint corresponds to the region of the plane that violates that given constraint, e.g., we notice that $x_1^\star(S)$ violates both constraints that belong to $\text{supp}(x_0^\star(S))$ and satisfies all the remaining ones. The result of Theorem 7 provides guarantees for the probability of violation of $x_2^\star(S)$. Note that the dashed-blue constraint is removed at stage 1, but it is not violated by the final solution of our scheme.

5.3.2 The non-degenerate case

In this subsection, we assume that problem P_k, $k \in \{0, \dots, \ell\}$, is non-degenerate.

Assumption 11 (Non-degeneracy). *For all $k \in \{0, \dots, \ell\}$, P_k is non-degenerate with $\mathbb{P}^m$-probability one.*

In case of a non fully-supported problem ($\text{supp}(x_k^\star(S)) < d$, for some $k \in \{0, \dots, \ell\}$), we adopt a procedure called regularization, in the same spirit as in [22]. This is based on introducing a lexicographic order as a tie-break rule to select which additional scenarios to append to $\text{supp}(x_k^\star(S))$, thus constructing a set of cardinality d. Note that unless we impose such an order there is no unique choice as all scenarios that are not included in $\text{supp}(x_k^\star(S))$ are not of support, hence their presence leaves the optimal solution unaltered. To this end, we put a unique linear order on the elements of S, i.e., we assign them a distinct numerical label. For each $k \in \{0, \dots, \ell\}$, let $\nu_k(S) = d - |\text{supp}(x_k^\star(S))|$ and define the following sets recursively as

$$
\begin{aligned}
Z_k(S) = \Big\{ &\nu_k(S) \text{ scenarios with the smallest labels in} \\
&S \setminus \Big(\bigcup_{j=0}^{k-1} \Big\{ \text{supp}(x_j^\star(S)) \cup Z_j(S) \Big\} \cup \text{supp}(x_k^\star(S)) \Big) \Big\},
\end{aligned} \tag{5.7}
$$

with $Z_0(S)$ containing the $\nu_0(S)$ smallest according to the linear order elements of $S \setminus \text{supp}(x_0^\star(S))$. Note that the set appearing in the definition of $Z_k(S)$ in (5.7) corresponds to scenarios available at stage k that are not of support.

For each $k \in \{0, \dots, \ell - 1\}$, we can now define the sets of discarded samples as

$$
R_k(S) = \text{supp}(x_k^\star(S)) \cup Z_k(S). \tag{5.8}
$$

Notice that by construction $|R_k(S)| = d$, while if for any $k \in \{0, \dots, \ell\}$, P_k is fully-supported, then $R_k(S) = \text{supp}(x_k^\star(S))$, i.e., it coincides with the support set of $x_k^\star(S)$. Similar to the fully-supported case, we denote by $R_\ell(S)$ the superset of

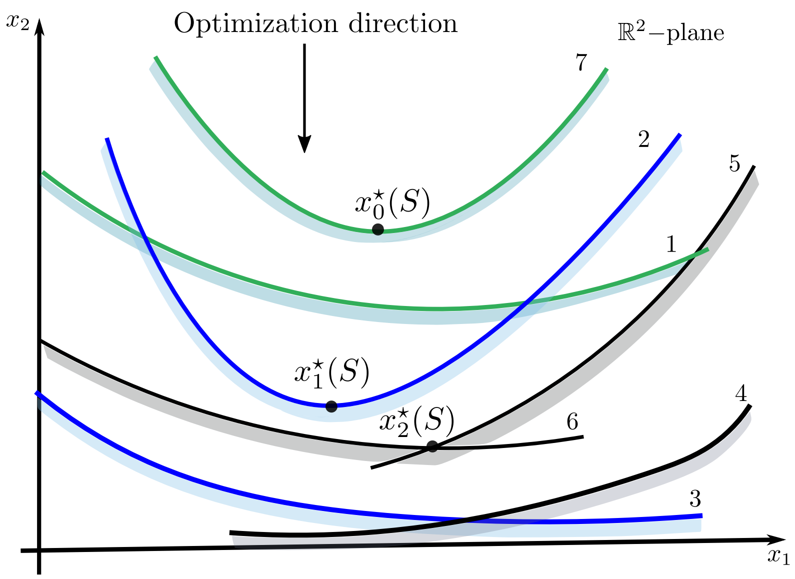


Figure 5.3: Illustration of the proposed scheme applied to non-degenerate, but not fully-supported, problems. The intermediate solutions are denoted by $x_k^\star(S)$, $k = 0, 1, 2$. The different colour patterns depict the removed scenarios at each stage. The green constraints are the ones removed at stage 0, the blue ones those removed at stage 1 of the scheme presented in Figure 5.1. Similar as before, the objective function is given by $c^\top x = x_2$ and this is indicated by the downwards arrow. Observe that the optimal solution, consequently the final solution returned by our scheme, depends on the linear order imposed to the original scenarios.

the support set of $x_\ell^\star(S)$ obtained by appending, if necessary, $\nu_\ell(S)$ scenarios from the remaining ones.

Notice that similarly to the fully-supported case, the objective function will necessarily decrease at each stage and we do not require that all scenarios discarded up to stage $k-1$, i.e., $\bigcup_{j=0}^{k-1} R_j(S)$, are violated by $x_k^\star(S)$.

Remark 3. *Consider two arbitrary scenario sets $C \subset C'$, and denote by $x_k^\star(C)$ and $x_k^\star(C')$ the minimizers of P_k with C and C', respectively, replacing S. Moreover, define $Z_k(C)$ and $Z_k(C')$ as in (5.7) with C and C', respectively, in place of S. The linear order imposed on S can be used to induce a lexicographic order on the cost of the intermidiate problems P_k by means of a process called regularization. In fact, by using the linear order on S, we may impose that $F_k(x_k^\star(C)) < F_k(x_k^\star(C'))$ if: either $c^\top x_k^\star(C) < c^\top x_k^\star(C')$; or $c^\top x_k^\star(C) = c^\top x_k^\star(C')$ and, at the first element that $Z_k(C)$ and $Z_k(C')$ differ, the corresponding label of $Z_k(C)$ is strictly lower with respect to the imposed linear order on S than the one of $Z_k(C')$. Moreover, it is shown in [22]* *that P_k with its objective function replaced by*

$$F_k(x) = \Big(c^\top x, Z_k(S)\Big) \tag{5.9}$$

is a fully-supported program, and the constructed set $R_k(S)$ in (5.8) forms its unique support set of cardinality d. Regularization is thus a way to select among subsets of scenarios that would otherwise yield the same objective value. We will use this procedure in Section 5.4.2 to prove Theorem 8 below.

We are now in position to state the main result related to non-degenerate problems.

Theorem 8. *Consider Assumptions 9 and 11. Fix $\epsilon \in (0,1)$, set $r = \ell d$ and let $m > r + d$. Consider also the scenario discarding scheme as encoded by (5.8) and illustrated in Figure 5.1, and let the minimizer of the ℓ-th program be $x^\star(S) = x^\star_\ell(S)$. We then have that*

$$\mathbb{P}^m\Big\{S \in \Delta^m : \mathbb{P}\Big\{\delta \in \Delta : g(x^\star(S), \delta) > 0\Big\} > \epsilon\Big\} \leq \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}. \tag{5.10}$$

It is important to note that (5.10) holds for any linear order imposed in the original samples S. Note that the optimal objective value of the scheme, however, depends on the imposed linear order, and we only provide feasibility guarantees and not optimality. However, this is also the case for the greedy strategy in [22], [30]. The only available results for the optimal cost are given in [30] when the removal scheme is the optimal one, which is, however, of combinatorial complexity.

Remark 4. *It should be noted that the assumption in [22], [30] appearing in the statement of Theorem 6, that requires all discarded scenarios to be violated by the final solution with $\mathbb{P}^m$-probability one, has some non-degeneracy implications for all intermediate problems. To see this, notice that if we allow for degenerate problems, then pathological situations where all scenarios are identical are admissible and may happen with non-zero probability (e.g., allowing for atomic masses). Clearly, in such* *cases there is no scenario that can be discarded while being violated by the resulting solution which remains unaltered. Therefore, we tighten the bound in Theorem 8, without strengthening the assumptions in [22], [30].*

To clarify how the scheme presented in Figure 5.1 works when applied to non-degenerate problems, consider the example depicted in Figure 5.3. Similar as before, we have $d = 2, m = 7$, and want to remove 4 constraints, i.e., $r = 4$. As opposed to Figure 5.2, however, note that the constraints are enumerated according to an arbitrary order, which is used to compose the sets $Z_k(S)$, $k \in \{0, 1, 2\}$, as described by Equation (5.7). Moreover, problems P_0 and P_1 are not fully-supported, as the number of support scenarios is equal to one in each of these cases. Our scheme first removes the scenario that supports the solution $x_0^\star(S)$ and the one labeled as 1, since it is the scenario with the smallest order among the remaining ones. These scenarios are depicted as green in Figure 5.3. Then, we solve problem P_1 with the resulting scenarios, obtaining $x_1^\star(S)$ as an intermediate solution and scenarios labeled as 2 and 3 to be removed. The former constraint is removed as it is in the support set of $x_1^\star(S)$, and the latter as it is the sample with the smallest index from the remaining ones. Finally, the solution provided by the scheme, and whose guarantees are given in Theorem 8, is denoted by $x_2^\star(S)$.

5.4 Proof of the main Results

The proofs of Theorems 7 and 8 are presented in the sequel. Even though Theorem 7 could have been obtained as a special case of Theorem 8, we decide to provide seperate arguments as the proof of the former is simpler and contains the main ideas behind our approach.

In both theorems, our argument proceeds as follows. We first show that there exists a natural mapping associated with the proposed removal procedure that takes as input the set of samples and returns a subset of Δ (similar to the discussion in Section 2.4). Then we show that such a mapping possesses a unique compression set. This latter step for the non-degenerate case involves the use of regularisation, which complicates and obscures the analysis. Our analysis is concluded by leveraging Theorem 1 to provide the proposed PAC bound on the probability of constraint violation.

5.4.1 The fully-supported case

Throughout this subsection, we consider Assumption 10. Let $m > (\ell+1)d$, and consider any set $C \subset S$, with $|C| = (\ell+1)d$. We consider the proposed scheme of Figure 5.1, fed by C rather than S. All quantities introduced in Section 5.3 depending on S would now depend on C instead. For a given set of scenarios $I \subset C$, we define

$$\begin{aligned} z^\star(I) = \operatorname*{argmin}_{x \in X} \quad & c^\top x \\ \text{subject to} \quad & g(x,\delta) \leq 0, \text{ for all } \delta \in I. \end{aligned} \tag{5.11}$$

Recall that $x_k^\star(C)$ denotes the minimizer of P_k which in turn is based on the samples in $C \setminus \cup_{j=0}^{k-1} R_j(C)$, i.e., the ones that have not been removed up to stage k of the proposed scheme. It thus holds that $x_k^\star(C) = z^\star(C \setminus \cup_{j=0}^{k-1} R_j(C))$ – note that the argument of $z^\star$ in this case depends on k, $k \in \{0,\dots,\ell\}$. Recall also that, under Assumption 10, we have $R_k(C) = \text{supp}(x_k^\star(C))$.

Since we will be invoking the framework introduced in Section 2.4, Chapter 2, we define the mapping $\mathcal{A} : \Delta^m \to 2^\Delta$, with $\zeta = (\ell+1)d$, as

$$\begin{aligned} \mathcal{A}(C) = & \Bigg\{ \Big\{ \delta \in \Delta : g(x_\ell^\star(C),\delta) \leq 0 \Big\} \\ & \cap \Bigg\{ \bigcap_{k=0}^{\ell} \Big\{ \delta \in \Delta : c^\top z^\star(J \cup \{\delta\}) \leq c^\top x_k^\star(C), \\ & \quad \text{for all } J \subset C \setminus \cup_{j=0}^{k-1} R_j(C), \text{ with } |J| = d-1 \Big\} \Bigg\} \Bigg\} \\ & \cup \Bigg\{ \bigcup_{k=0}^{\ell-1} R_k(C) \Bigg\} = \Big(\mathcal{A}_1(C) \cap \mathcal{A}_2(C) \Big) \cup \mathcal{A}_3(C). \end{aligned} \tag{5.12}$$

The main motivation to define the mapping in (5.12) is the fact that its probability of violation will be shown to upper bound that of $\{\delta \in \Delta : g(x_\ell^\star(C),\delta) \leq 0\}$, which is ultimately the quantity we are interested in (as shown in Section 5.4.1, step 3). It is worth pointing out that the mapping $\mathcal{A}$ in (5.12) is defined for any finite set of samples (similar to the mapping in Remark 1 of Section 2.4). We decided not to index it with the cardinality of its domain to ease notation.

Note that $\mathcal{A}(C)$ comprises three sets:

i $\mathcal{A}_1(C)$ contains all realisations of δ for which the final decision of our proposed scheme $x^\star_\ell(C) = x^\star(C)$ remains feasible. This is the set whose probability of occurrence we are ultimately interested to bound.

ii $\mathcal{A}_2(C)$, the intersection of $\ell + 1$ sets indexed by $k \in \{0, \dots, \ell\}$, each of them containing the realisations of δ such that, for all subsets of cardinality $d-1$ from the remaining samples at stage k, the cost $c^\top z^\star(J \cup \{\delta\})$ is lower than or equal to $c^\top x^\star_k(C)$. The former cost corresponds to appending δ to any set J of $d-1$ scenarios from $C \setminus \cup_{j=0}^{k-1} R_j(C)$, while the latter corresponds to the cost of the minimizer $x^\star_k(C)$ of P_k. Informally, this inequality is of similar nature with that of the first set in $\mathcal{A}(C)$, however, rather than considering constraint satisfaction it only involves some cost dominance condition for each of the interim and the final optimal solutions. The motivation to use this representation rather than constraint satisfaction conditions stems from the fact that in Section 5.3.2 we will be appending a lexicographic order to the cost so that we break the tie among multiple compression sets. Besides, these sets carry information about the path taken by the proposed scheme, which is to be understood, in this context, as the sequence $(x_k(C))_{k=0}^{\ell}$.

iii $\mathcal{A}_3(C)$, which includes all scenarios that are removed by the discarding scheme. Implicit in the definition of mapping (5.12) is the fact that, for any compression set C, all samples that are not removed in the intermediate stages must be contained in the set $\mathcal{A}_1(C) \cap \mathcal{A}_2(C)$. This fact will be crucial in the following arguments.

The following proposition establishes a basic property of any compression associated to the mapping (5.12).

Proposition 8. *Consider Assumptions 9 and 10. Set $r = \ell d$ and let $m > (\ell+1)d$. We have that $C \subset S$ is a compression set for $\mathcal{A}(C)$ in* (5.12) *if and only if, for all $k \in \{0,\dots,\ell\}$,*

$$x_k^\star(C) = x_k^\star(S). \tag{5.13}$$

Proof. We first show necessity. Suppose that C is a compression set but, for the sake of contradiction, we have that there exists $k \in \{0,\dots,\ell\}$ and $\bar{\delta} \in S \setminus C$ such that

$$x_k^\star(C) \neq x_k^\star(C \cup \{\bar{\delta}\}). \tag{5.14}$$

Let $\bar{k}$ be the minimum index such that (5.14) holds, while we have that $x_j^\star(C) = x_j^\star(C \cup \{\bar{\delta}\})$, for all $j < \bar{k}$.

By Assumption 10, the last statement implies that $\mathrm{supp}(x_j^\star(C)) = \mathrm{supp}(x_j^\star(C \cup \{\bar{\delta}\}))$, for all $j < \bar{k}$, as the support set of each optimal solution is unique. Hence, $R_j(C) = R_j(C \cup \{\bar{\delta}\})$ for all $j < \bar{k}$, and $R_j(C) = \mathrm{supp}(x_j^\star(C))$ for fully-supported problems (similarly for $R_j(C \cup \{\bar{\delta}\})$). By (5.11), we then have

$$x_{\bar{k}}^\star(C) = z^\star(C \setminus \cup_{j=0}^{\bar{k}-1} R_j(C)), \tag{5.15}$$

$$x_{\bar{k}}^\star(C \cup \{\bar{\delta}\}) = z^\star((C \setminus \cup_{j=0}^{\bar{k}-1} R_j(C)) \cup \{\bar{\delta}\}). \tag{5.16}$$

Since the right-hand side of (5.16) involves one more scenario with respect to the right-hand side of (5.15), the feasible set of (5.16) is a subset set of the one of (5.15). Moreover, by the fact that $x_{\bar{k}}^\star(C \cup \{\bar{\delta}\}) \neq x_{\bar{k}}^\star(C)$ and Assumption 9, we obtain that

$$c^\top x_{\bar{k}}^\star(C) < c^\top x_{\bar{k}}^\star(C \cup \{\bar{\delta}\}). \tag{5.17}$$

Notice that $\bar{\delta}$ belongs to the support set of $x_{\bar{k}}^\star(C \cup \{\bar{\delta}\})$, as its removal results in a different optimal solution with lower cost in (5.17). In other words, there exists $\bar{J} \subset C \setminus \cup_{j=0}^{\bar{k}-1} R_j(C)$ (in fact, $\bar{J} = \mathrm{supp}(x_{\bar{k}}^\star(C \cup \{\bar{\delta}\})) \setminus \{\bar{\delta}\}$) of cardinality $d-1$ such that by (5.16), we have that

$$c^\top x_{\bar{k}}^\star(C) < c^\top x_{\bar{k}}^\star(C \cup \{\bar{\delta}\}) = c^\top z^\star(\bar{J} \cup \{\bar{\delta}\}). \tag{5.18}$$

At the same time, C is assumed to be a compression set. Since $\bar{\delta} \notin C$, then $\bar{\delta} \notin \cup_{k=0}^{\ell-1} R_k(C) = \mathcal{A}_3(C)$, as $\cup_{k=0}^{\ell-1} R_k(C) \subset C$. As a result, $\bar{\delta}$ will give rise to a constraint in P_ℓ, hence $\bar{\delta} \in \mathcal{A}_2(C)$, which in turn implies that for all $J \subset C \setminus \cup_{j=0}^{\ell-1} R_j(C)$ with $|J| = d-1$, and for all $k \leq \ell$,

$$c^\top z^\star(J \cup \{\bar{\delta}\}) \leq c^\top x^\star(C) \leq c^\top x_k^\star(C), \tag{5.19}$$

where the first inequality follows from the fact that $c^\top x^\star(C)$ is the optimal value for P_ℓ, and $x^\star(C) = x_\ell^\star(C)$ by construction satisfies all constraints with scenarios in $J \cup \{\bar{\delta}\}$. The second inequality follows from the fact that $k \leq \ell$, and the cost deteriorates as k increases. Setting $k = \bar{k}$ and $J = \bar{J}$ in (5.19) establishes a contradiction with (5.18), thus showing that $x_k^\star(C) = x_k^\star(C \cup \{\delta\})$, for any $\delta \in S \setminus C$, and any $k \in \{0, \ldots, \ell\}$. Proceeding inductively, adding one by one each element in $S \setminus C$, we can show that $x_k^\star(C) = x_k^\star(S)$, for any $k \in \{0, \ldots, \ell\}$, thus concluding the necessity part of the proof.

We now show sufficiency. Let $C \subset S$ be such that $x_k^\star(C) = x_k^\star(S)$ for all $k \in \{0, \ldots, \ell\}$. We aim to show that C is a compression for S, i.e., $\delta \in \mathcal{A}(C)$ for all $\delta \in S$. Recalling the definition of the mapping $\mathcal{A}(C)$ from (5.12) we note that, under this scenario, the sets $\mathcal{A}_1(C)$ and $\mathcal{A}_3(C)$ are trivially equal to $\mathcal{A}_1(S)$ and $\mathcal{A}_3(S)$, respectively. Moreover, since $C \subset S$ and $x_k^\star(C) = x_k^\star(S)$ for all $k \in \{0, \ldots, \ell\}$, which implies that $R_k(C) = R_k(S)$ by Assumption 9, we have that $S \setminus \cup_{j=0}^{k-1} R_j(S) = S \setminus \cup_{j=0}^{k-1} R_j(C) \supset C \setminus \cup_{j=0}^{k-1} R_j(C)$. The latter implies then that the inequalities in $\mathcal{A}_2(S)$ constitute a superset of those in $\mathcal{A}_2(C)$, hence, that problem is more constrained and as a result $\mathcal{A}_2(S) \subset \mathcal{A}_2(C)$. By construction we have that $\delta \in \mathcal{A}(S)$ for all $\delta \in S$. This in turn implies that if a sample is not removed, then it will have to be included in $\mathcal{A}_2(S)$, and due to the established inclusion also in $\mathcal{A}_2(C)$. Since $\mathcal{A}_1(S) = \mathcal{A}_1(C)$ and $\mathcal{A}_3(S) = \mathcal{A}_3(C)$, we then have that $\delta \in \mathcal{A}(C)$ for all $\delta \in S$, showing that C is a compression set. This concludes the proof of the proposition. $\square$

A natural compression candidate is

$$C = \bigcup_{k=0}^{\ell} \mathrm{supp}(x_k^\star(S)), \tag{5.20}$$

as it consists of the support sets of the intermediate problems.

Existence: We prove that C in (5.20) is a compression set. By the sufficient part of Proposition 8, it suffices to show that the set C in (5.20) satisfies $x_k^\star(C) = x_k^\star(S)$, for all $k \in \{0, \dots, \ell\}$. We will show this by means of induction. For the base case $k = 0$, notice that

$$c^\top x_0^\star(S) = c^\top z^\star(S) = c^\top z^\star(\mathrm{supp}(x_0^\star(S))) = c^\top x_0^\star(C), \tag{5.21}$$

where the first equality is due to (5.11), the second equality is due to the fact that $\mathrm{supp}(x_0^\star(S))$ is the support set of $x_0^\star(S)$, while the last equality is due to Assumption 10, the definition of support set and the fact that $\mathrm{supp}(x_0^\star(S)) \subset C$. By (5.21), and Assumption 9, we conclude that $x_0^\star(C) = x_0^\star(S)$.

To complete the induction argument, we assume that $x_j^\star(C) = x_j^\star(S)$ for all $j \in \{0, \dots, \bar{k}\}$, for some $\bar{k} < \ell$. We will show that $x_{\bar{k}+1}^\star(C) = x_{\bar{k}+1}^\star(S)$. To this end, by Assumption 10, $x_j^\star(C) = x_j^\star(S)$ for all $j \leq \bar{k}$ implies that $\mathrm{supp}(x_j^\star(C)) = \mathrm{supp}(x_j^\star(S))$, for all $j \leq \bar{k}$, as the support set of each optimal solution is unique. Moreover, $R_j(C) = R_j(S)$ for all $j < \bar{k}$, as $R_j(C) = \mathrm{supp}(x_j^\star(C))$ for fully-supported problems. Similarly to the base case we have that

$$c^\top x_{\bar{k}+1}^\star(C) = c^\top z^\star(C \setminus \cup_{j=0}^{\bar{k}} R_j(S)) \leq c^\top z^\star(S \setminus \cup_{j=0}^{\bar{k}} R_j(S)) = c^\top x_{\bar{k}+1}^\star(S), \tag{5.22}$$

where the first and last equalities are due to (5.11), and the inequality is due to the fact that $C \setminus \cup_{j=0}^{\bar{k}} R_j(S) \subseteq S \setminus \cup_{j=0}^{\bar{k}} R_j(S)$. Moreover

$$\begin{aligned} c^\top x_{\bar{k}+1}^\star(S) &= c^\top z^\star(S \setminus \cup_{j=0}^{\bar{k}} R_j(S)) \\ &= c^\top z^\star(\mathrm{supp}(x_{\bar{k}+1}^\star(S))) \leq c^\top z^\star(C \setminus \cup_{j=0}^{\bar{k}} R_j(S)) = c^\top x_{\bar{k}+1}^\star(C), \end{aligned} \tag{5.23}$$

where the first and last equalities are due to (5.11), the second one due to the fact that $\mathrm{supp}(x_{\bar{k}+1}^\star(S)) \subset S \setminus \cup_{j=0}^{\bar{k}} R_j(S)$, and the inequality holds since $R_j(C) = R_j(S)$ and

$\text{supp}(x^\star_{\bar{k}+1}(S)) \subset C \setminus \cup_{j=0}^{\bar{k}} R_j(S)$. By (5.22) and (5.23) we then have that $x^\star_{\bar{k}+1}(C) = x^\star_{\bar{k}+1}(S)$, thus concluding the induction proof. In other words, we have shown that

$$x^\star_k(C) = x^\star_k(S), \text{ for all } k \in \{0, \ldots, \ell\}. \tag{5.24}$$

Relation (5.24) together with the sufficiency part of Proposition 8 shows that the candidate C in (5.20) is a compression set.

Uniqueness: To show that C in (5.20) is the unique compression set, assume for the sake of contradiction that there exists another compression $C' \subset S$ for the mapping defined in (5.12), $C' \neq C$, with $|C'| = (\ell + 1)d$. Since $C' \subset S$ is a compression, Proposition 8 (necessity part) implies that $x^\star_k(C') = x^\star_k(S)$, for all $k \in \{0, \ldots, \ell\}$, as C' is a compression. Besides, by the existence part (Step 1 above), we have shown that for C given in (5.20) we have that $x^\star_k(C) = x^\star_k(S)$ for all $k \in \{0, \ldots, \ell\}$. We thus have that for all $k \in \{0, \ldots, \ell\}$, $x^\star_k(C) = x^\star_k(C')$. This in turn implies that $\text{supp}(x^\star_k(C)) = \text{supp}(x^\star_k(C'))$ for all $k \in \{0, \ldots, \ell\}$, which, by Assumption 10, leads to $C = C'$ (to see this notice that $\cup_{k=0}^{\ell} \text{supp}(x^\star_k(S)) \subset C'$ and $|C'| = (\ell + 1)d$), thus establishing a contradiction.

Linking Theorem 1 with the probability of constraint violation: Recall that

$$\mathcal{A}(C) = \Big(\mathcal{A}_1(C) \cap \mathcal{A}_2(C)\Big) \cup \mathcal{A}_3(C), \tag{5.25}$$

where the individual sets are as in (5.12). Recall also that $\mathcal{A}_3(S)$ is a discrete set that contains the removed samples throughout the execution of the scheme of Figure 5.1. Fix any S with m scenarios, set $r = \ell d$ and let $m > (\ell+1)d$. Fix also $\epsilon \in (0, 1)$. Let $C \subset S$ with $|C| = (\ell+1)d$ be the unique compression defined in (5.20). We have that

$$\begin{aligned} \mathbb{P}\{\mathcal{A}(C)\} &= \mathbb{P}\{(\mathcal{A}_1(C) \cap \mathcal{A}_2(C)) \cup \mathcal{A}_3(C)\} = \mathbb{P}\{\mathcal{A}_1(C) \cap \mathcal{A}_2(C)\} \le \mathbb{P}\{\mathcal{A}_1(C)\} \\ &= \mathbb{P}\{\delta \in \Delta : \ g(x^\star(C), \delta) \le 0\}, = \mathbb{P}\{\delta \in \Delta : \ g(x^\star(S), \delta) \le 0\}, \end{aligned} \tag{5.26}$$

where the first equality is due to the fact that $\mathbb{P}\{\mathcal{A}_3(C)\} = 0$, since $\mathcal{A}_3(C)$ is a discrete set and we have imposed the non-degeneracy condition of Assumption 11 which prevents scenarios to have accumulation points with non-zero probability, while the inequality is due to the fact that $\mathcal{A}_1(C) \cap \mathcal{A}_2(C) \subseteq \mathcal{A}_1(C)$. The second last equality is by definition of $\mathcal{A}_1(C)$, and the last one follows from the fact that $x^\star(C) = x^\star(S)$ (see (5.24)).

We then have that if $\mathbb{P}\{\delta \in \Delta : \ g(x^\star(S), \delta) > 0\} > \epsilon$ then $\mathbb{P}\{\delta \in \Delta : \ \delta \notin \mathcal{A}(C)\} > \epsilon$. As a result, $\{S \in \Delta^m : \ \mathbb{P}\{\delta \in \Delta : \ g(x^\star(S), \delta) > 0\} > \epsilon\} \subseteq \{S \in \Delta^m : \ \mathbb{P}\{\delta \in \Delta : \ \delta \notin \mathcal{A}(C)\} > \epsilon\}$. The last statement implies then that

$$\begin{aligned} &\mathbb{P}^m\{S \in \Delta^m : \ \mathbb{P}\{\delta \in \Delta : \ g(x^\star(S), \delta) > 0\} > \epsilon\} \\ &\leq \mathbb{P}^m\{S \in \Delta^m : \ \mathbb{P}\{\delta \notin \mathcal{A}(C)\} > \epsilon\}. \end{aligned} \tag{5.27}$$

Therefore, since set C in (5.20) is the unique compression of $\mathcal{A}(C)$, by Theorem 1 we have that

$$\mathbb{P}^m\{S \in \Delta^m : \ \mathbb{P}\{\delta \in \Delta : \ \delta \notin \mathcal{A}(C)\} > \epsilon\} \leq \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}. \tag{5.28}$$

By (5.27) and (5.28) we then have that $\mathbb{P}^m\{(\delta_1, \dots, \delta_m) \in \Delta^m : \ \mathbb{P}\{\delta \in \Delta : \ g(x^\star(S), \delta) > 0\} > \epsilon\} \leq \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}$, thus concluding the proof of Theorem 7. □

5.4.2 The non-degenerate case

Throughout this subsection, we consider Assumption 11. Let $m > (\ell+1)d$, and consider any set $C \subset S$ with $|C| = (\ell+1)d$. We modify the mapping $\mathcal{A}(C)$ in (5.12) by replacing the second set in its definition with

$$\begin{aligned} \mathcal{A}_2(C) = \bigcap_{k=0}^{\ell} \Big\{ &\delta \in \Delta : F_k(z^\star(J \cup \{\delta\})) \leq F_k(x_k^\star(C)), \\ &\text{for all } J \subset C \setminus \cup_{j=0}^{k-1} R_j(C), \text{ with } |J| = d-1 \Big\}, \end{aligned} \tag{5.29}$$

where $F_k(\cdot)$ is the augmented objective function defined in (5.9), related to P_k defined by means of the regularization procedure of Section 5.3. The above inequality is to be understood in a lexicographic sense as detailed in Remark 3. A natural candidate compression set in this case is

$$C = \bigcup_{k=0}^{\ell} (\text{supp}(x_k^\star(S)) \cup Z_k(S)), \tag{5.30}$$

which is composed by the removed samples of the scheme, and the support set of the last stage together with the corresponding constraints in $Z_\ell(S)$. In fact, we now append $Z_k(S)$ in the definition of C to ensure that $|C| = (\ell+1)d$, as $|\text{supp}(x_k^\star(S))|$ could be lower than d as the intermediate problems might not be fully-supported. Similarly to the fully-supported case, our goal is to show that the compression set defined in (5.30) is the unique compression set of size $(\ell+1)d$ for the mapping in (5.12), with $A_2(C)$ in (5.30) in place of $\mathcal{A}_2(C)$ in (5.12). By (5.8), recall that $R_k(C) = \text{supp}(x_k^\star(C)) \cup Z_k(C),\ k \in \{0,\dots,\ell-1\}$.

Proposition 9. *Suppose Assumptions 9 and 11 hold. Let C be the set in* (5.30)*, and consider the scheme of Figure 5.1 with the removed scenarios given by* (5.8)*. We have that:*

i) $x_k^\star(C) = x_k^\star(S)$ *and* $Z_k(C) = Z_k(S)$ *for all* $k \in \{0,\dots,\ell\}$.

ii) *Let C' be any other compression of size $(\ell+1)d$. Suppose $R_j(C) = R_j(C')$ for all $j \in \{0,\dots,\bar{k}-1\}$, where $\bar{k}$ is the smallest index such that $x_{\bar{k}}^\star(C') \neq x_{\bar{k}}^\star(C)$. Then $x_{\bar{k}}^\star(C') \neq x_{\bar{k}}^\star(C' \cup \{\delta\})$ for some $\delta \in \text{supp}(x_{\bar{k}}^\star(C)) \setminus \text{supp}(x_{\bar{k}}^\star(C'))$. Moreover, such a δ is in fact in the set $C \setminus C'$.*

Proof. Item i): We use induction. Fix $k = 0$ and note that

$$x_0^\star(C) = z^\star(C) = z^\star(\text{supp}(x_0^\star(S))) = x_0^\star(S), \tag{5.31}$$

where the first equality follows from the definition in (5.11), for the second one we use the definition of the support set, and the third one follows from the definition of $x_0^\star(S)$ and the definition of the support set. Moreover, we have that

$$\begin{aligned} Z_0(C) &= \Big\{\nu_0(S) \text{ scenarios with the smallest labels in } C \setminus \big\{\text{supp}(x_0^\star(S))\big\}\Big\} \\ &= \Big\{\nu_0(S) \text{ scenarios with the smallest labels in } S \setminus \big\{\text{supp}(x_0^\star(S))\big\}\Big\} \\ &= Z_0(S), \end{aligned} \tag{5.32}$$

where the first equality is due to the definition of C in (5.30) and the fact that $Z_0(S) \subset C$, while the last one is due to the definition of $Z_0(S)$ in (5.7). Assume now that $x^\star_k(C) = x^\star_k(S)$ and $Z^\star_k(C) = Z^\star_k(S)$ for all $k \in \{0, \dots, \bar{k}\}$, and consider the case $\bar{k}+1$. Indeed, we have that

$$\begin{aligned} x^\star_{\bar{k}+1}(C) &= z^\star(C \setminus \cup_{j=0}^{\bar{k}} R_j(C)) = z^\star(\mathrm{supp}(x^\star_{\bar{k}+1}(S))) \\ &= z^\star(S \setminus \cup_{j=0}^{\bar{k}} R_j(S)) = x^\star_{\bar{k}+1}(S), \end{aligned} \tag{5.33}$$

where these relations follow as in (5.31) for the case $k = 0$. We also have that

$$\begin{aligned} Z^\star_{\bar{k}+1}(C) = \Big\{&\nu_{\bar{k}+1}(S) \text{ scenarios with the smallest labels in} \\ &C \setminus \Big(\bigcup_{j=0}^{\bar{k}} R_j(S) \cup \mathrm{supp}(x^\star_{\bar{k}+1}(S)) \Big) \Big\} = Z^\star_{\bar{k}+1}(S), \end{aligned} \tag{5.34}$$

since $Z^\star_{\bar{k}+1}(S) \subset C \setminus \bigcup_{j=0}^{\bar{k}} \{R_j(S) \cup \mathrm{supp}(x^\star_{\bar{k}+1}(S))\}$ due to the particular choice of C in (5.30), thus proving that for C in (5.30) we have $x^\star_k(C) = x^\star_k(S)$ and $Z_k(C) = Z_k(S)$, for all $k \in \{0, \dots, \ell\}$. This concludes the proof of item i).

Item ii): We prove the contrapositive. Assume that for all $\delta \in \mathrm{supp}(x^\star_{\bar{k}}(C)) \setminus \mathrm{supp}(x^\star_{\bar{k}}(C'))$ we have that $x^\star_{\bar{k}}(C') = x^\star_{\bar{k}}(C' \cup \{\delta\})$. We will show that $x^\star_{\bar{k}}(C) = x^\star_{\bar{k}}(C')$. We then have that

$$c^\top x^\star_{\bar{k}}(C') = c^\top x^\star_{\bar{k}}(C' \cup \{\delta\}) = c^\top x^\star_{\bar{k}}(C' \cup \mathrm{supp}(x^\star_{\bar{k}}(C))) = c^\top x^\star_{\bar{k}}(C), \tag{5.35}$$

where the second equality holds due to Lemma 2.12 in [22] since $C' \cup \{\delta\} \subset C' \cup \mathrm{supp}(x^\star_{\bar{k}}(C))$. The last equality follows from the definition of the support set and the non-degeneracy condition of Assumption 3. By Assumption 9 we then conclude that $x^\star_{\bar{k}}(C) = x^\star_{\bar{k}}(C')$.

We now show that such a δ must belong to $C \setminus C'$. In fact, choose $\bar{\delta} \in \mathrm{supp}(x^\star_{\bar{k}}(C)) \setminus \mathrm{supp}(x^\star_{\bar{k}}(C'))$ and assume for the sake of contradiction that $\bar{\delta} \in C'$. This implies that $\bar{\delta} \in R_j(C')$ for some $j \geq \bar{k}$. In this is the case, we have that

$$\begin{aligned} c^\top x^\star_{\bar{k}}(C' \cup \{\bar{\delta}\}) &= c^\top z^\star\Big(\Big(C' \setminus \bigcup_{j=0}^{\bar{k}-1} R_j(C)\Big) \cup \{\bar{\delta}\}\Big) \\ &= c^\top z^\star \left(\mathrm{supp}(x^\star_{\bar{k}}(C'))\right) = c^\top x^\star_{\bar{k}}(C') \end{aligned} \tag{5.36}$$

where the first relation holds due to (5.11) and the fact that $R_j(C') = R_j(C)$ for all $j < \bar{k}$, the second one is due to the fact that $\mathrm{supp}(x^\star_{\bar{k}}(C')) \subset C' \setminus \bigcup_{j=0}^{\bar{k}-1} R_j(C) \cup \{\bar{\delta}\}$

and $\bar{\delta} \in R_j(C')$ for $j \geq \bar{k}$. The third equality follows from the definition of the support set and the non-degeneracy condition of Assumption 11. However, note that (5.36) contradicts our choice of $\bar{\delta}$, which requires that $x^\star_{\bar{k}}(C') \neq x^\star_{\bar{k}}(C' \cup \{\bar{\delta}\})$. This concludes the proof. $\square$

Proof of Theorem 8: Existence. The existence part follows *mutatis mutandis* from the one of Theorem 7. In fact, $\mathcal{A}_1(C) = \mathcal{A}_1(S)$ and $\mathcal{A}_3(C) = \mathcal{A}_3(S)$ by Proposition 9, item *i*), and $\mathcal{A}_2(S) \subset \mathcal{A}_2(C)$ as $C \subset S$ (see the discussion at the end of Proposition 8).

Uniqueness: Let C' be another compression of size $(\ell+1)d$ and assume for the sake of contradiction that $C \neq C'$. We can distinguish two possible cases. Case I: there exists a $\bar{k} \in \{0, \dots, \ell\}$ such that $x^\star_{\bar{k}}(C') \neq x^\star_{\bar{k}}(C)$; or case II: $x^\star_k(C') = x^\star_k(C)$ for all $k \in \{0, \dots, \ell\}$, but there exists a $\tilde{k} \in \{0, \dots, \ell\}$ such that $Z_{\tilde{k}}(C') \neq Z_{\tilde{k}}(C)$. In the sequel, we argue separately that neither of these cases can happen.

Case I: Let $\bar{k}$ be the smallest index such that $x^\star_{\bar{k}}(C') \neq x^\star_{\bar{k}}(C)$, and let $\tilde{k} \leq \bar{k}$ be the smallest index such that $Z_{\tilde{k}}(C') \neq Z_{\tilde{k}}(C)$. Consider first the case where $\tilde{k} < \bar{k}$. Under these definitions, note that $R_j(C') = R_j(C)$ for all $j < \tilde{k}$. Moreover, we have that

$$\begin{aligned} Z_{\tilde{k}}(C') &= \Big\{ \nu_{\tilde{k}}(C) \text{ scenarios with the smallest labels in } C' \setminus \bigcup_{j=0}^{\tilde{k}-1} R_j(C') \Big\} \\ &= \Big\{ \nu_{\tilde{k}}(C) \text{ scenarios with the smallest labels in } C' \setminus \bigcup_{j=0}^{\tilde{k}-1} R_j(S) \Big\}, \end{aligned} \tag{5.37}$$

where the first equality is by the definition in (5.7) and the fact that $\nu_{\tilde{k}}(C') = \nu_{\tilde{k}}(C)$ since $\tilde{k} < \bar{k}$; the second equality follows since $R_j(C') = R_j(C) = R_j(S)$ – the last equality follows from Proposition 9, item *i*) – for all $j \leq \tilde{k} - 1$ and due to the uniqueness requirement of Assumption 9. Note that $Z_{\tilde{k}}(C') \neq Z_{\tilde{k}}(S)$ and $C' \subset S$ implies that, for all $\delta \in Z_{\tilde{k}}(C') \setminus Z_{\tilde{k}}(S)$,

$$y_\delta > \max_{\xi \in Z_{\tilde{k}}(S)} y_\xi = y_{\max}, \tag{5.38}$$

where $y_\delta \in \mathbb{N}$ corresponds to the label associated to δ.

We will use the relation (5.38) to show that any element in $C \setminus C'$ has a label greater than $y_{\max}$. In fact, note that

$$C' \setminus C \subset \left\{\cup_{j=\tilde{k}+1}^{\ell} R_j(C')\right\} \cup \{Z_{\tilde{k}}(C') \setminus Z_{\tilde{k}}(C)\}, \tag{5.39}$$

hence it suffices to show that any element in either set in the right-hand side of (5.39) is greater than $y_{\max}$. To this end, fix any $\delta \in \cup_{j=\tilde{k}+1}^{\ell} R_j(C')$ and note that

$$y_\delta > \max_{\xi \in Z_{\tilde{k}}(C') \setminus Z_{\tilde{k}}(C)} y_\xi > y_{\max}, \tag{5.40}$$

where the first inequality is due to the fact that since such a δ has not been removed up to stage $\tilde{k}$, then its label will be greater than the ones in $Z_{\tilde{k}}(C')$, and as a result the ones in $Z_{\tilde{k}}(C') \setminus Z_{\tilde{k}}(C)$. The second inequality follows from (5.38) and the fact that $Z_{\tilde{k}}(C') \setminus Z_{\tilde{k}}(C) \subset Z_{\tilde{k}}(S)$. Therefore, for any $\delta \in C' \setminus C$ we have that $y_\delta > y_{\max}$.

From now on, let δ be the scenario associated to $y_{\max}$. Pick $\bar{J} = \{\text{supp}(x^\star_{\tilde{k}}(C))\} \cup \{Z_{\tilde{k}}(C) \setminus \{\delta\}\}$, which has cardinality $d-1$ and is a subset of $C \setminus \cup_{j=0}^{\tilde{k}-1} R_j(C)$, and fix $\bar{\delta} \in C' \setminus C$. Note that under this choice of $\bar{\delta}$

$$F_{\tilde{k}}(z^\star(\bar{J} \cup \{\bar{\delta}\})) > F_{\tilde{k}}(x^\star_{\tilde{k}}(C)), \tag{5.41}$$

since $y_{\bar{\delta}} > y_{\max}$ (by our previous discussion) and the inequality is interpreted lexicographically. However, this contradicts the fact that C is a compression set (see Definition 15) as $\bar{\delta} \in C' \setminus C \subset S$, hence $\bar{\delta} \notin \mathcal{A}_3(C')$ has not been removed, but $\bar{\delta} \notin \mathcal{A}_2(C)$ due to (5.41).

Consider now the case $\tilde{k} = \bar{k}$. Note that, in this case, we have that $R_j(C') = R_j(C)$ for all $j \leq \bar{k} - 1$. Based on the result of Proposition 9, item *ii*), applied to C' (note that the assumptions of Proposition 9, item *ii*), are satisfied with our choice of C'), we observe that there exists a $\bar{\delta} \in \{\text{supp}(x^\star_{\bar{k}}(C)) \setminus \text{supp}(x^\star_{\bar{k}}(C'))\} \cap \{C \setminus C'\}$ such that $x^\star_{\bar{k}}(C') \neq x^\star_{\bar{k}}(C' \cup \{\bar{\delta}\})$. Repeating the arguments following Equations (5.15) and (5.16) in the necessity proof of Proposition 8 with C' in the place of C in that proposition, we reach a contradiction that C' is a compression set.

Case II: We can reach a contradiction if case II holds in a similar fashion as in case I. In fact, letting $\tilde{k}$ be the smallest index such that $Z_{\tilde{k}}(C') \neq Z_{\tilde{k}}(C)$, the proof proceeds in an identical manner with case I.

Hence, we conclude that in any case $C = C'$, thus proving uniqueness of the compression set in (5.30).

Linking Theorem 1 with the probability of violation: Note that for the non-degenerate case the mapping has the same structure as the one in (5.12), with the set $\mathcal{A}_2(C)$ in (5.12) being substituted with the one in (5.29). The arguments then follows *mutatis mutandis* the ones in the last part of the fully-supported case. This concludes the proof of Theorem 8. □

5.5 Tightness of the bound of Theorem 7

5.5.1 Class of programs for which the bound is tight

We provide a sufficient condition on the problems P_k so that the solution returned by the scheme of Figure 5.1 achieves the upper bound given by the right-hand side of (5.10) when all the intermediate problems P_k, $k = 0, \dots, \ell$, are fully-supported. The result of this section implies that the bound of Theorem 7 is tight, i.e., there exists a class of convex scenario programs where it holds with equality.

To this end, we replace the mapping $\mathcal{A}$ in (5.12) with $\bar{\mathcal{A}} : \Delta^m \to 2^\Delta$ defined

$$\bar{\mathcal{A}}(C) = \Big\{ \delta \in \Delta : g(x_\ell^\star(C), \delta) \leq 0 \Big\} \cup \Big\{ \bigcup_{k=0}^{\ell-1} \mathrm{supp}(x_k^\star(C)) \Big\}. \tag{5.42}$$

Note that $\bar{\mathcal{A}}(C)$ coincides with the one in (5.25), but without the set $\mathcal{A}_2(C)$ in its definition. We impose the following assumption.

Assumption 12. *Fix any $S = \{\delta_1, \dots, \delta_m\} \in \Delta^m$ and let $C \subset S$. For any $k \in \{0, \dots, \ell\}$ and $\delta \in S$ such that $\delta \in \mathrm{supp}(x_k^\star(C))$, we have that*

$$g(z^\star(J), \delta) > 0,$$

for all $J \subset C \setminus \Big(\cup_{j=0}^{k-1} \mathrm{supp}(x_j^\star(C)) \cup \{\delta\} \Big)$ with $|J| = d$.

Assumption 12 imposes a restriction on the class of fully-supported problems. For instance, the pictorial example of Figure 5.2 does not satisfy Assumption 12, even though all the intermediate problems P_k are fully-supported, as the dashed-blue removed constraint is not violated by the resulting solution. Indeed, Assumption 12 requires that, with $\mathbb{P}^m$-probability one, whenever a sample belongs to the support scenarios of any intermediate problem, then the scenario associated with it is violated by all the solutions that could have been obtained using any subset of cardinality d from the remaining samples. Note that verifying Assumption 4 is hard in general; we show in the next subsection an example that satisfies this requirement and admits an analytic solution. Assumption 12 is similar to the requirement of Theorem 6 [22], [30], however, in Theorem 9 below we exploit it in conjunction with the discarding scheme of Figure 5.1 to show that the result of Theorem 7 is tight. This serves as a constructive argument for the existential result of [30].

Note that in this chapter we do not offer any means to check the validity of Assumption 12; however, we show that this class of problems is not empty in the next section.

Theorem 9. *Consider Assumptions 9, 10, and 12. Fix $\epsilon \in (0,1)$, set $r = \ell d$ and let $m > r + d$. Consider also the scenario discarding scheme as encoded by (5.5) and illustrated in Figure 5.1, and let the minimizer of the ℓ-th program be $x^\star(S) = x^\star_\ell(S)$. We then have that*

$$\mathbb{P}^m\Big\{S \in \Delta^m : \mathbb{P}\Big\{\delta \in \Delta : g(x^\star(S),\delta) > 0\Big\} > \epsilon\Big\} = \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}. \quad (5.43)$$

Proof. Existence: We first show that the set C given in (5.20) is a compression for the mapping in (5.42). Recall that under Assumption 10 we have that $R_k(S) = \text{supp}(x^\star_k(S))$ for all $k \in \{0,\dots,\ell\}$. Applying a similar induction argument as in the existence part of Theorem 7, we have that $x^\star_k(C) = x^\star_k(S)$ for all $k \in \{0,\dots,\ell\}$. Hence, by the definition of the mapping $\bar{\mathcal{A}}(C)$ in (5.42), we obtain that $\bar{\mathcal{A}}(C) = \bar{\mathcal{A}}(S)$, thus showing that C in (5.20) is a compression.

Uniqueness: Let C' be another compression of size $(\ell+1)d$. We will show that $x^\star_k(C') = x^\star_k(S)$ for all $k \in \{0,\dots,\ell\}$, which by the existence part yields that $x^\star_k(C) = x^\star(C')$ for all $k \in \{0,\dots,\ell\}$. By Assumption 9 and 10, this would then imply that $C = C'$.

To show that $x_k^\star(C') = x_k^\star(S)$ for all $k \in \{0, \dots, m\}$, it suffices to show that for all $\delta \in S \setminus C'$ we have that

$$x_k^\star(C') = x_k^\star(C' \cup \{\delta\}), \text{ for all } k \in \{0, \dots, \ell\}. \tag{5.44}$$

In fact, if (5.44) holds for all $\delta \in S \setminus C'$ by induction it follows then that $x_k^\star(C') = x_k^\star(S)$ for all $k \in \{0, \dots, \ell\}$.

To show (5.44) assume for the sake of contradiction that there exist a $\bar{\delta} \in S \setminus C'$ and a $k \in \{0, \dots, \ell\}$ such that $x_k^\star(C) \neq x_k^\star(C' \cup \{\bar{\delta}\})$. Let $\bar{k}$ be the smallest index such that this occurs and note that

$$x_{\bar{k}}^\star(C') = z^\star(C' \setminus \cup_{j=0}^{\bar{k}-1} \text{supp}(x_j^\star(C'))), \tag{5.45}$$

$$x_{\bar{k}}^\star(C' \cup \{\bar{\delta}\}) = z^\star((C' \setminus \cup_{j=0}^{\bar{k}-1} \text{supp}(x_j^\star(C')) \cup \{\bar{\delta}\}), \tag{5.46}$$

which implies that $\bar{\delta} \in \text{supp}(x_{\bar{k}}^\star(C' \cup \{\bar{\delta}\}))$, as removal of $\bar{\delta}$ will change $x_{\bar{k}}^\star(C' \cup \{\bar{\delta}\})$ to $x_{\bar{k}}^\star(C')$. By Assumption 12 and since $\text{supp}(x_j^\star(C')) = \text{supp}(x_j^\star(C' \cup \{\bar{\delta}\}))$ for all $j = 0, \dots, \bar{k}-1$, we have that for all $J \subset C' \setminus \left(\cup_{j=0}^{\bar{k}-1} \text{supp}(x_j^\star(C')) \cup \{\bar{\delta}\} \right)$ with cardinality d,

$$g(z(J), \bar{\delta}) > 0. \tag{5.47}$$

Hence, since $\bar{J} = \text{supp}(x_\ell^\star(C'))$ is a subset of cardinality d of $C' \setminus \left(\cup_{j=0}^{\bar{k}-1} \text{supp}(x_j^\star(C')) \cup \{\bar{\delta}\} \right)$, as these constraints have not been removed from C', we obtain that

$$g(z(\bar{J}), \bar{\delta}) = g(x_\ell^\star(C'), \bar{\delta}) > 0, \tag{5.48}$$

where the equality follows from (5.11). However, C' is assumed to be a compression set for $\bar{\mathcal{A}}$, which implies that $\delta \in \bar{\mathcal{A}}(C')$, i.e., $g(x_\ell^\star(C'), \bar{\delta}) \leq 0$. This is in contradiction with (5.48), implying that $x_k^\star(C') = x_k^\star(C' \cup \delta)$, for any $k \in \{0, \dots, \ell\}$, for any $\delta \in S \setminus C'$. Using induction, adding one by one $\delta \in S \setminus C'$, we can then show that $x_k^\star(C') = x_k^\star(S) = x_k^\star(C)$ for all $k \in \{0, \dots, \ell\}$, thus showing that C in (5.20) is the unique compression set for the mapping defined in (5.42).

By Theorem 1, we then have that

$$\begin{aligned} &\mathbb{P}^m\{S \in \Delta^m : \mathbb{P}\{\delta : \delta \notin \bar{\mathcal{A}}(C)\} > \epsilon\} = \mathbb{P}^m\{S \in \Delta^m : \mathbb{P}\{\delta : g(x_\ell^\star(C), \delta) > 0\} > \epsilon\} \\ &= \mathbb{P}^m\{S \in \Delta^m : \mathbb{P}\{\delta : g(x_\ell^\star(S), \delta) > 0\} > \epsilon\} = \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}, \end{aligned} \tag{5.49}$$

where the first equality follows since the union of support scenarios is a discrete set and will be of measure zero, since the problems are assumed to be fully-supported, and hence non-degenerate. To obtain the second equality we have used the fact that $x^\star_\ell(C) = x^\star_\ell(S)$ for the compression set defined in (5.20). This concludes the proof of Theorem 9. □

5.5.2 An example with an analytic solution

We revisit the problem studied in [28], [30] and show that it satisfies Assumption 12. We compute analytically the violation probability of the solution returned by applying the scheme of Figure 5.1 to this problem and show that the resulting violation probability is in line with the result of Theorem 9.

To this end, fix $m \in \mathbb{N}$ and $r < m$, and consider the procedure of Section 5.3, which involves a sequence of $\ell + 1$ problems. For $k = 0, \ldots, \ell$, each of them in the form of P_k, and is given by

$$\begin{aligned} \underset{x\in[0,1]}{\text{minimise}} \quad & x \\ \text{subject to} \quad & x \geq \delta_i, \quad i \in S \setminus \cup_{j=0}^{k-1} R_j(S). \end{aligned} \tag{5.50}$$

We further assume that all samples are extracted from a uniform distribution over the interval $[0, 1]$. Note that (5.50) satisfies Assumptions 9 and 10. Also notice that Assumption 4 is satisfied for this problem, as $x^\star_k = \max_{i\in S\setminus \cup_{j=0}^{k-1} R_j(S)} \delta_i$, i.e., the maximum among the scenarios available at stage k of the discarding process. Under the choice of a uniform distribution, the support set is a singleton, i.e., the maximizing scenario is unique, with $\mathbb{P}^m$-probability one. Therefore, once the single support scenario is removed, the new minimizer will necessarily be lower thus violating the removed scenario. Note that since this is an one-dimensional problem $(d = 1)$, our procedure involves removing samples one by one.

Let $\epsilon \in (0, 1)$ and $r = \ell d = \ell < m$ (since this is an one-dimensional problem), and consider the sets:

$$B = \{S \in \Delta^m : x^\star(S) < 1 - \epsilon\}, \tag{5.51}$$

which represents the measure of samples such that the probability of constraint violation is greater that ϵ, as $\mathbb{P}\{\delta \in \Delta : x^\star(S) < \delta\} = 1 - x^\star(S)$ due to the fact that $\mathbb{P}$ is uniform on $[0, 1]$. Moreover,

$$A_0 = \{S \in \Delta^m : \text{ for all } i = 1, \dots, m, \delta_i \leq 1 - \epsilon\}, \tag{5.52}$$

and for all $\ell = 1, \dots, m$,

$$A_i = \{S \in \Delta^m : \text{ there exist exactly } \\ i \text{ samples greater than } 1 - \epsilon\}. \tag{5.53}$$

Observe that $\{A_i\}_{i=0}^m$ forms a partition of Δ^m, and notice that

$$\mathbb{P}^m\{A_0\} = \mathbb{P}^m\{S \in \Delta^m : \delta_i \leq 1 - \epsilon, \text{ for all } \\ i = 1, \dots, m\} = (1 - \epsilon)^m, \tag{5.54}$$

where the last equality is due to sample independence. Since $\mathbb{P}^m\{\delta_i \in \Delta : \delta_i > 1 - \epsilon\} = \epsilon$ (due to the fact that $\mathbb{P}$ is uniform on $[0, 1]$) and A_i involves i independent samples, we have that

$$\mathbb{P}^m\{A_i\} = \binom{m}{i} \epsilon^i, \tag{5.55}$$

where the factor $\binom{m}{i}$ accounts for all combinations of i out of m samples. Moreover, for $i \leq r$, $\mathbb{P}^m\{B | A_i\} = (1 - \epsilon)^{m-i}$ since it involves conditioning on exactly i samples being removed, and computing the probability that the returned solution $x^\star(S)$ is feasible for the remaining $m - i$ samples. By the total law of probability, we then have that

$$\mathbb{P}^m\{S \in \Delta^m : x^\star(S) < 1 - \epsilon\} = \mathbb{P}^m\{B\} = \sum_{i=0}^{m} \mathbb{P}^m\{A_i \cap B\} \\ = \sum_{i=0}^{r} \mathbb{P}^m\{A_i \cap B\} + \sum_{i=r+1}^{m} \mathbb{P}^m\{A_i \cap B\}. \tag{5.56}$$

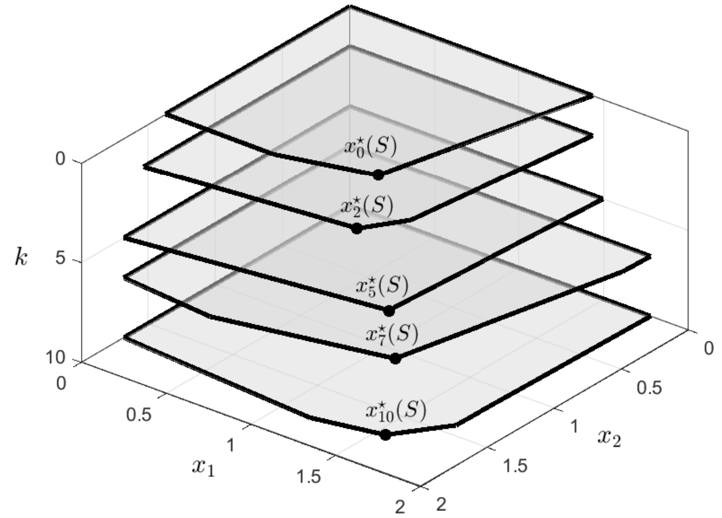


Figure 5.4: Feasibility sets of the intermediate problems P_k, $k = 0, 2, 5, 7$, 10, for the scheme proposed in Figure 5.1 when applied to (5.58). The optimal solution of each problem is denoted by $x_k^\star(S)$, $k = 0, 2, 5, 7$, 10.

The terms involving $i > r$ are zero because $(\delta_1, \ldots, \delta_m) \in B$ implies that at most r of samples are greater than $1 - \epsilon$, i.e., $\mathbb{P}^m\{B|A_i\} = 0$, for all $i > r$. Hence,

$$\mathbb{P}^m\{S \in \Delta^m : x^\star(S) < 1 - \epsilon\} = \sum_{i=0}^{r} \mathbb{P}^m\{A_i \cap B\}$$
$$= \sum_{i=0}^{r} \mathbb{P}^m\{B|A_i\}\mathbb{P}^m\{A_i\} = \sum_{i=0}^{r} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}, \tag{5.57}$$

where the last equality follows from (5.54) and (5.55). Note that this result coincides with the bound of Theorem 9.

5.6 Numerical example

In this section, we consider a resource allocation problem to illustrate our theoretical result. Suppose that a manufacturer produces a good in d different locations, and that this good can be produced from n different resources. The quantity of resource p, $p = 1, \ldots, n$, that is needed to produce a unitary amount of the given good at facility j, $j = 1, \ldots, d$, is a random variable parametrized by $\delta \in \mathbb{R}$, and is denoted by $a_{pj}(\delta)$. We assume that the amount of resources p available to all facilities is deterministic. The objective is to maximize the production, given by $\sum_{j=1}^{d} x^j$, where x^j is the $j-$th component of $x \in \mathbb{R}^d$, while keeping the risk of running out of resources under control.

Under the scenario theory we do not have access to the distribution that generates $a_{pj}(\delta), p = 1, \ldots, n, j = 1, \ldots, d$; however, we encode it by means of data $(a_{pj}(\delta_i))_{i=1}^m$ for all $p = 1, \ldots, n$ and for all $j = 1, \ldots, d$, and solve the following convex scenario problem

$$\begin{aligned} &\underset{\{x^j \geq 0\}_{j=1}^d}{\text{minimise}} \quad c^\top x \\ &\text{subject to} \quad A(\delta_i)x \leq b, \quad \text{for all } i = 1, \ldots, m, \end{aligned} \tag{5.58}$$

where, for each $i \in \{1, \ldots, m\}$, $A(\delta_i) \in \mathbb{R}^{n \times d}$ is a matrix whose (p, j)-th entry is given by $a_{pj}(\delta_i)$, $b \in \mathbb{R}^n$ is a vector whose $p-$th component is the amount of resource p available to all facilities, and $c = \begin{bmatrix} -1 & \ldots & -1 \end{bmatrix}^\top \in \mathbb{R}^d$.

Set $d = 2$ and consider 2000 scenarios from the unknown distribution[1] for δ. We study the behavior of the scheme in Figure 5.1 when we discard $r = 20$ of these scenarios. In this case, note that according to the description given in Section 5.3, we have to solve a cascade of 11 optimisation problems (i.e, $\ell = 10$ in the scheme of Figure 5.1).

Figure 5.4 illustrates the feasible set for stages $k = 0, 2, 5, 7$, and 10 of the scheme of Figure 5.1, and depicts the corresponding optimal solution for each P_k as $x_k^\star(S)$. Note that the feasible set associated to each problem P_k grows as we remove scenarios. To complement this analysis, we also show in Figure 5.5 a comparison between our method and the greedy scenario removal strategy as described in [30], which removes scenarios one by one according to one that yields the best improvement in the cost. With the blue dots we show the cost obtained by the proposed procedure, where we are allowed to remove scenarios in batches of $d = 2$, while the solid one shows the performance obtained by the greedy removal strategy,

[1]For our simulations, fix $i \in \{1, \ldots, m\}$ and generate an auxiliary matrix, $B(\delta_i) \in \mathbb{R}^{n \times d}$, whose entries are obtained from a Laplacian distribution with mean equal to one and variance equal to three. Then set $A(\delta_i) = 0.04B(\delta_i)$. Our numerical results were obtained setting the "seed" equal to 30 in MATLAB.

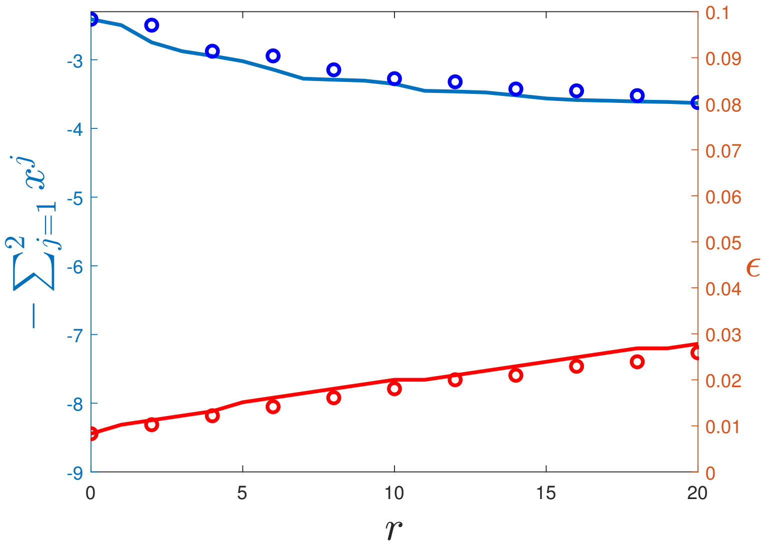


Figure 5.5: Cost and probability of constraint violation for the solution returned by the scheme of Figure 5.1 and a greedy removal strategy for the problem in (5.58) when $d = 2$, $n = 2$, and $m = 2000$. With the blue dots we show the cost obtained by the proposed procedure, where we are allowed to remove scenarios in batches of $d = 2$, while the solid one shows the performance obtained by the greedy removal strategy where scenarios are removed one by one. In red we show the corresponding behavior of the probability of constraint violation. This is calculated from the bounds of Theorem 4 and Theorem 1, respectively, using numerical inversion and $\beta = 1 \times 10^{-6}$.

where scenarios are removed one by one. In red we show the corresponding behavior of the probability of constraint violation. This is calculated from the bounds of Theorem 7 and Theorem 6, respectively, using numerical inversion and $\beta = 1 \times 10^{-6}$.

Consider now (5.58) with $d = 10$ and the same 2000 scenarios. We compare the cost improvement of the proposed bound (Theorem 8) with the one of Theorem 6 [30]. To this end, for a given $\epsilon \in [0.01, 0.08]$, we compute the maximum number of scenarios that can be removed using each of these bounds. Note that due to the fact that we remove scenarios is batches of d, we compute the number of scenarios that need to be removed by means of numerical inversion from the bound of Theorem 4 (using $m = 2000$, $\beta = 1 \times 10^{-6}$ and the given ϵ), and round it down to the closest multiple of $d = 10$. For instance, for $\epsilon = 0.03$ the maximum number of scenarios that can be removed using the bound in (8) is $r = 18$, but we only remove 10. Figure 5.6 shows then the relative cost difference $100 \times \frac{f^\star(\epsilon) - \bar{f}^\star(\epsilon)}{f^\star(\epsilon)}$ as a function of ϵ, where $f^\star(\epsilon)$ is the optimal value of problem (5.58) when scenarios are removed according

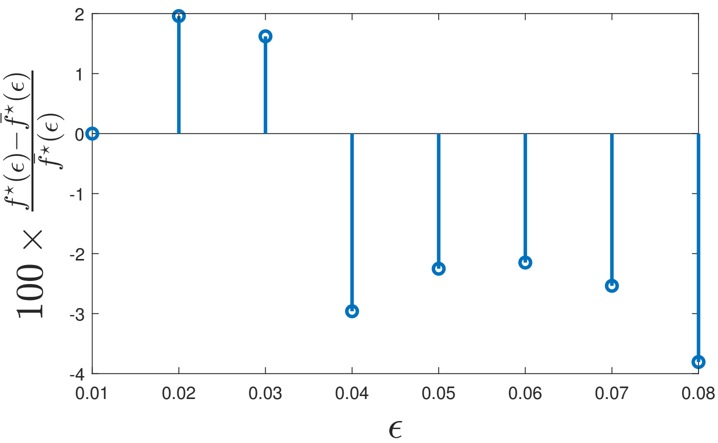


Figure 5.6: Relative cost improvement $100 \times \frac{f^\star(\epsilon)-\bar{f}^\star(\epsilon)}{f^\star(\epsilon)}$, as a function of ϵ, where $f^\star(\epsilon)$ corresponds to the cost associated with Theorem 8, and $\bar{f}^\star(\epsilon)$ to the one of Theorem 6, [30]. The numerical results correspond to (5.58) with $d = 10$.

to Theorem 8, and $\bar{f}^\star(\epsilon)$ correspond to the bound in [30]. Note that for $\epsilon \geq 0.03$ our scheme results in an improvement in performance achieving approximately 4% improvement when $\epsilon = 0.08$. This is due to the fact that more scenarios can be removed, while guaranteeing the same level of violation. Notice also that there is no improvement when $\epsilon \leq 0.03$. This is due to the limitation on the number of removed scenarios, as for $\epsilon \in [0.01, 0.03]$ the proposed bound returns a value for r that is less than 10, hence no scenarios are removed.

Note that the computational requirements of the proposed approach are lower with respect to the greedy removal strategy of [30] (see also [22]). To put this in perspective, to remove 100 scenarios in the previous example when $d = 10$, the greedy strategy, as described in [30], requires the solution of 1101 optimisation problems of the form (5.58), whereas the proposed scheme only needs to solve 11 of these problems. The computational savings are more pronounced as the dimension of the problem grows.

Remark 5. *Note that Theorem 8 offers an improvement with respect to Theorem 6, [22], [30], as far as probabilistic guarantees on the probability of constraint violation are concerned. However, the effect of the proposed discarding scheme with* *respect to the greedy removal strategy of [22], [30], does not necessarily follow the one of Figure 5.6, but is problem dependent.*

5.7 Conclusion

In this chapter we have addressed the third challenge stated in Chapter 1, namely, we have studied optimisation problems with uncertain constraints. We revisited the sampling and discarding approach for scenario based optimisation and proposed a scenario discarding scheme that consists of a cascade of optimisation problems, where at each stage we remove a superset of the support constraints. By relying on results from compression learning theory, we provided a tighter bound on the probability of constraint violation of the obtained solution, extending state-of-the-art bounds. Besides, we have shown that the proposed bound is tight, and characterized the class of problems for which this is the case.

The main limitation of the results of this chapter is that we require the number of removed scenarios to be an integer multiple of the dimension of the decision space. In the next chapter we extend the analysis of the removal scheme proposed in this chapter to account for the case where the number of removed scenarios is arbitrary.

6

Extension to the case of an arbitrary number of removed scenarios

In this chapter we continue our study on optimisation problems in the presence of uncertain constraints under the lens of the scenario approach theory. More specifically, we extend the analysis of the removal scheme proposed in Chapter 5 for the class of fully-supported scenario programs to the case where an arbitrary number of scenarios is removed, i.e., without requiring the number of discarded scenarios to be a multiple of the dimension of the decision space.

6.1 Introduction

Data abound in modern applications, and this can be leveraged to boost robustness against uncertainty. In the past decades, new research directions have sprung from this fact, and are now shaping the theoretical foundation of several disciplines, including control theory and machine learning. Under this scenario, several data-driven algorithms, such as the scenario approach theory, came to prominence.

Chapter 5 has studied a specific removal scheme and provided a bound on the probability of constraint violation that outperforms the one in [30]. It also shows tightness of the proposed bound by providing a class of scenario programs that achieves this bound with equality. The removal scheme analysed in Chapter 5 is composed by a cascade of scenario programs where at each stage a superset of the support scenarios associated to the optimal solution is removed. However, their analysis restricts the number of discarded scenarios to be a multiple of the dimension of the corresponding scenario program.

This chapter explores the extent to which the analysis of this particular removal scheme can be generalised to allow for arbitrary discarded scenarios. First, we characterise the class of scenario programs that permits such arbitrary removal without introducing additional conservatism in the feasibility bound. We show that this coincides with the class of problems that led to tight bounds in Chapter 5. For general scenario programs, we propose a more conservative feasibility bound which, however, improves upon the bound in [30]. Moreover, if we are dealing with a min-max scenario program, we also discuss an alternative removal algorithm that combines removal procedure proposed in Chapter 5 with the strategy presented in [35] and [57].

This chapter is organised as follows. In Section 6.2, we briefly review the removal procedure studied in Chapter 5. The extension of the results of Chapter 5 to an arbitrary number of discarded scenarios is presented in Section 6.3. In Subsection 6.3.1 we study such a generalisation for a subclass of fully-supported scenario programs, while in 6.3.2 we provide a more conservative bound that is valid to any fully-supported scenario programs. In the latter subsection, we also consider an adaptation of our procedure to min-max scenario programs. Section 6.4 summarises the results of this chapter. The proofs of some of the results of Section 6.3 are presented in Section 6.5.

6.2 Review of the removal procedure of Chapter 5

We deal with the sampling-and-discarding approach to scenario optimisation and the removal scheme proposed in Chapter 5. To this end, we consider the scenario program

$$\begin{aligned} \underset{x \in \mathcal{X}}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta_i) \leq 0, \ \text{for all } \delta_i \in S \setminus R(S), \end{aligned} \tag{6.1}$$

where the quantities involved in this problem have been defined in Section 5.2, with the samples $S = \{\delta_1, \dots, \delta_m\}$ being independently sampled from the unknown probability distribution $\mathbb{P}$. Throughout this chapter we consider the following assumption on (6.1).

Assumption 13. *Problem* (6.1) *is fully-supported*[1] *and its solution exists and is unique for any* $\{\delta_1, \dots, \delta_m\}$*. Moreover, its feasible set has a non-empty interior.*

We also consider that the samples in S are ordered, i.e., there exists a bijection $\sigma : \{1, \dots, m\} \to S$, and, for any $i, j \in \{1, \dots, m\}$, $i \neq j$, we say that δ_i is smaller than or equal to δ_j whenever $\sigma^{-1}(\delta_i) \leq \sigma^{-1}(\delta_j)$ in the usual sense. Strict inequalities can be used with a similar interpretation. For a fixed $S = \{\delta_1, \dots, \delta_m\}$, as in Chapter 5, we denote the optimal solution of a scenario program as in (6.1) for a generic $J \subset S$ as

$$\begin{aligned} z^\star(J) = \operatorname*{argmin}_{x \in \mathcal{X}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta) \leq 0, \text{ for all } \delta \in J. \end{aligned} \tag{6.2}$$

The samples $R(S)$ in (6.1) can be discarded through the procedure described in Chapter 5, which is now reviewed for convenience. Let $r < m$ be the number of discarded constraints and write $r = q_1 d + q_2$ using the division algorithm, where q_1 and q_2 are integers and $q_2 < d$. For $k \in \{0, \dots, q_1\}$, consider the sequence of $q_1 + 1$ scenario programs given by

$$\begin{aligned} P_k : \operatorname*{minimise}_{x \in \mathcal{X}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta) \leq 0, \text{ for all } \delta \in S \setminus R_k(S), \end{aligned} \tag{6.3}$$

where $R_0(S)$ is the empty set, $R_k(S) = R_{k-1}(S) \cup \operatorname{supp}(x^\star_{k-1}(S))$ for $k \in \{1, \dots, q_1\}$, with $x^\star_k(S)$, $k = 0, \dots, q_1$, representing[2] the optimal solution of (6.3). If q_2 is not equal to zero, we define similarly a scenario program P_{q_1+1} with $R_{q_1+1}(S) =$

[1] Most of the results in this chapter can be extended to non-fully-supported but non-degenerate problems (see [29] for the definition) using the same technique as in Section 5.3.2 of Chapter 5 (see also [22] and [124]), by ordering the samples and creating an augmented (regularised) optimisation problem.

[2] Note that q_1 here plays the role of ℓ in Chapter 5.

P_k	Removed till $k \in \{0, \ldots, q_1\}$	Optimiser at $(k+1)$-th stage
0	$R_0(S) = \emptyset$	$x_0^\star(S)$
1	$R_1(S) = \text{supp}(x_0^\star(S))$	$x_1^\star(S)$
$\vdots$	$\vdots$	$\vdots$
q_1	$R_{q_1}(S) = R_{q_1-1}(S) \cup \text{supp}(x_{q_1-1}^\star(S))$	$x_{q_1}^\star(S)$
$q_1 + 1$	$R_{q_1}(S) \cup \bar{R}(S)$	$x_{q_1+1}^\star(S)$

Table 6.1: Description of the quantities at the interim stages for the procedure encoded by (6.3).

$R_{q_1}(S) \cup \bar{R}(S)$, where $\bar{R}(S)$ is a subset of size q_2 from $\text{supp}(x_{q_1}^\star(S))$ containing the q_2-th smallest scenarios according to ordering defined by σ. As the scenario program P_k depends on the solution of the previous stage through $R_k(S)$, this removal scheme can be interpreted as a cascade of $q_1 + 2$ (or $q_1 + 1$, if q_2 is equal to zero) scenarios programs where at each stage the support set associated to the optimal solution is removed and possibly a subset of the support set in the last stage if $k = q_1$ and $q_2 \neq 0$. These quantities are summarised in Table 6.1. Let

$$x^\star(S) = \begin{cases} x_{q_1}^\star(S), & \text{if } q_2 = 0; \\ x_{q_1+1}^\star(S), & \text{if otherwise} . \end{cases} \tag{6.4}$$

Observe that $x^\star(S)$ is the final decision whose probability of constraint violation we are ultimately interested in.

If $q_2 = 0$, i.e., if the removed scenarios form an integer multiple of d ($r = q_1 d$), then the results in Chapter 5 allow assessing the probability of constraint violation of $x^\star(S)$. Recall also that Chapter 5 shows tightness of the bound by means of a class of problems that, roughly speaking, requires removed constraints to be violated by the optimal solution of any scenario program whose constraint are enforced using remaining scenarios. Indeed, Chapter 5 imposes Assumption 14, which is presented in the sequel for convenience.

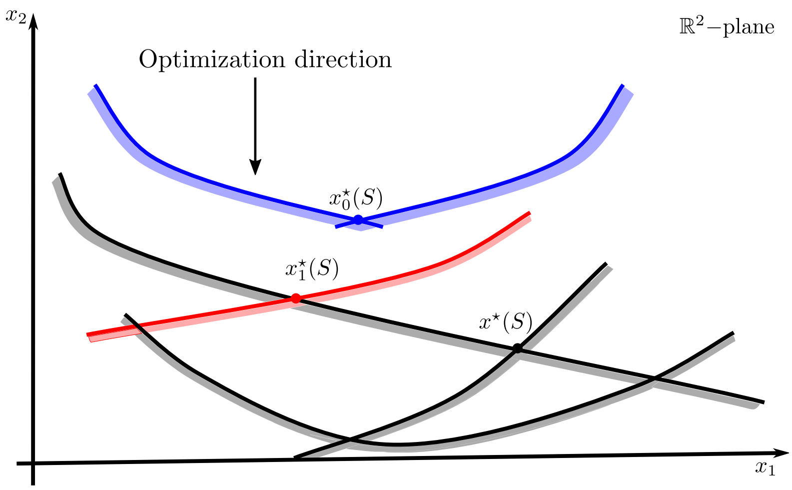
Assumption 14. *Let $S = \{\delta_1, \ldots, \delta_m\}$ be i.i.d. samples from the unknown probability distribution $\mathbb{P}$ and let $C \subset S$ be any subset of S. For any $k \in \{0, \ldots, q_1\}$ if $\delta \in \text{supp}(x_k^\star(C))$, then we have that*

$$g(z^\star(J), \delta) > 0, \text{ for all } J \subset C \setminus (\cup_{j=0}^{k-1} \text{supp}(x_j^\star(C)) \cup \{\delta\}).$$

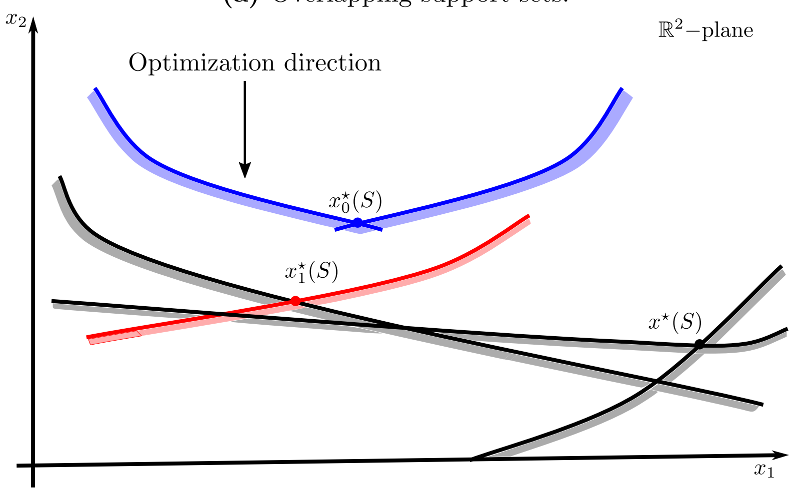
6.3 Main results

Throughout this section we will explore the removal strategy described by (6.3) when the number of discarded scenarios is not an integer multiple of the dimension of the optimisation problem, i.e., when $q_2 \neq 0$. We first show that this is not a straightforward generalisation of the analysis presented in Chapter 5, as it entails certain difficulties. To this end, consider two realisations of a 2-dimensional ($d = 2$) scenario program as depicted in Figure 6.1. In both of these realisations our goal is to remove three of the six samples, i.e., we have $q_1 = 1$ and $q_2 = 1$.

We focus first on the realisation shown in Figure 6.1a. Following the procedure described in (6.3), the 1st stage removes the scenarios highlighted in blue, as these compose the support set of $x_0^\star(S)$. To remove the third scenario we solve the corresponding scenario program without the scenarios highlighted in blue and obtain $x_1^\star(S)$ as the optimal solution. Assume that the ordering (as detailed in Section 6.2) is such that the scenario highlighted in red is discarded, then we obtain the solution depicted in $x^\star(S)$. For this realisation, the set composed by the two blue scenarios, the red scenario, and the support set of $x^\star(S)$ constitutes a subset of the samples with cardinality equal to $r + d = 3 + 2 = 5$ such that following the same procedure using only these 5 samples we would obtain the same solutions. Informally, this is related to the notion of compression set introduced in Section 2.4, Chapter 2. Unfortunately, this conclusion is sample dependent and does not hold uniformly across all the samples. For instance, consider the realisation illustrated in Figure 6.1b. The removal algorithm described in (6.3) proceeds similarly as in the previous case; however, we notice that the final decision is supported by two scenarios that do not belong to the support set of the previous iteration. This latter fact implies that the cardinality of the subset of the samples that would lead to the same solutions with those that would have been obtained if all the samples were employed is no longer 5 but 6. The difference between these two instances is that in the first one the support sets associated to the two last stages overlap, while in the second one these are disjoint. Moreover, the smaller the cardinality of the corresponding compression set, the tighter the bound one can offer. In view of a



(a) Overlapping support sets.



(b) Disjoint support sets.

Figure 6.1: Two different realisations (6.1a and 6.1b) of a two dimensional ($d = 2$) scenario program with six scenarios ($m = 6$) from which three scenarios are discarded ($r = 3$). The scenarios highlighted in blue represent the support set of the 1st stage of the removal procedure, and the ones in red the scenarios removed in the 2nd stage. In realisation 6.1a the scenario that has not been removed in the 2nd stage belongs to the support set of $x^\star(S)$, which is the optimal solution of the 3rd stage, while in the realisation in 6.1b none of the remaining scenarios from the 2nd stage belong to the support set of $x^\star(S)$.

tight bound, this observation motivates restricting attention to the class of problems where the one of Figure 6.1a belongs. We formalise this in the next section.

6.3.1 Arbitrary number of removed scenarios under Assumption 14.

Inspired by the discussion in the previous section, the most natural direction if one wants to produce a tight bound on the resulting decision is that of preventing the situation of Figure 6.1b to happen. The main result of this section is to reveal that this can be obtained by means of the Assumption 14, which has been exploited in Chapter 5 to obtain Theorem 9. To this end, the following proposition is instrumental.

Proposition 10. *Consider the removal procedure encoded by* (6.3)*. Let* $S \in \{\delta_1, \dots, \delta_m\}$ *be i.i.d. samples from the unknown probability distribution* $\mathbb{P}$*, and* $r = q_1 d + q_2$*, with* $0 < q_2 < d$*. Under Assumptions 13 and 14, if* δ *is a scenario in* $\mathrm{supp}(x^\star_{q_1}(S))$ *that has not been removed in the* (q_1+1)*-th stage, i.e.,* $\delta \in \mathrm{supp}(x^\star_{q_1}(S)) \setminus \bar{R}(S)$*, then* δ *is in the support set of* $\mathrm{supp}(x^\star(S))$.

Proof. Consider the P_{q_1+1} that would arise if $q_2 \neq 0$ and recall that, by (6.4) , $x^\star(S) = x^\star_{q_1+1}(S)$. Recall also that $R_{q_1+1}(S) = R_{q_1}(S) \cup \bar{R}(S)$, where $\bar{R}(S)$ contains the q_2-th smallest scenarios of $\mathrm{supp}(x^\star_{q_1}(S))$ that will be removed at the (q_1+1)-th stage. With this in mind, let us prove this proposition by contradiction.

Suppose there exists $\bar{\delta} \in \mathrm{supp}(x^\star_{q_1}(S)) \setminus \bar{R}(S)$ that is not of support for $x^\star(S)$. Such a $\bar{\delta}$ is feasible for problem P_{q_1+1}, i.e., we must have that $g(x^\star(S), \bar{\delta}) \leq 0$. Choose $\bar{J} = \mathrm{supp}(x^\star(S)) \subset S \setminus \{R_{q_1}(S) \cup \{\bar{\delta}\}\}$ (due to the fact that $\bar{\delta} \notin \mathrm{supp}(x^\star(S))$), which then implies that $z^\star(\bar{J}) = x^\star(S)$ and $g(z^\star(\bar{J}), \bar{\delta}) \leq 0$. Under Assumption 14, with $k = q_1$ and since $R_{q_1}(S) = \cup_{j=0}^{q_1-1} \mathrm{supp}(x^\star_j(S))$, the latter is a contradiction, since this would require $g(z^\star(J), \bar{\delta}) > 0$. This concludes the proof of the proposition. □

In other words, Proposition 10 shows that under Assumption 14 the realisation of Figure 6.1b can only happen with probability zero. Proposition 10 will be used as the main step to extend the results of Chapter 5 for the subclass of scenario programs that satisfy Assumptions 13 and 14. To achieve this, we leverage the results presented in Section 2.4, Chapter 2, by relying on the concept of compression to establish a probably approximately correct (PAC) bound on the probability of constraint violation as in Theorem 1.

In the sequel, we identify, under Assumption 14, how the analysis carried in Chapter 5 can be extended to encompass an arbitrary number of discarded scenarios. We consider the removal strategy described in the previous section. Since all the intermediate problems are fully-supported we remove at each stage the associated support set and in the (q_1+1)-th stage only a subset of the support set is removed, if q_2 is not zero. Define $\mathcal{A} : \Delta^m \to 2^\Delta$ as

$$\mathcal{A}(C) = \{\delta \in \Delta : g(x^\star(C), \delta) \le 0\} \cup \left\{ \bigcup_{j=0}^{q_1-1} \mathrm{supp}(x_j^\star(C)) \cup \bigcup_{\delta \in \bar{R}(C)} \delta \right\}, \tag{6.5}$$

which contains the discarded scenarios in the discrete set, and the set we are ultimately interested in, namely, the set $\{\delta \in \Delta : g(x^\star(C), \delta) \le 0\}$. Following the algorithmic description presented in (6.3) a candidate compression set is given by

$$C = \bigcup_{j=0}^{q_1} \mathrm{supp}(x_j^\star(S)) \cup \mathrm{supp}(x^\star(S)), \tag{6.6}$$

as it contains, under Assumption 13, all the support sets associated to the scenario programs P_k, $k \in \{0, \dots, q_1+1\}$ (due to the fact that $q_2 \neq 0$ and Proposition 10 holds).

Remark 6. *By Proposition 10, we have that* $\mathrm{supp}(x_{q_1}^\star(S)) \cup \mathrm{supp}(x^\star(S)) = \bar{R}(S) \cup \mathrm{supp}(x^\star(S))$, *as any* $\delta \in \mathrm{supp}(x_{q_1}^\star(S))$ *but not in* $\bar{R}(S)$ *will be in* $\mathrm{supp}(x^\star(S))$. *As such,* $|C| = r + d$ *as opposed to* $(q_1+2)d$.

Proposition 11. *Consider the removal procedure described by* (6.3). *Under Assumptions 13 and 14, the set in* (6.6) *is the unique compression set of size* $r+d$ *associated to the mapping* (6.5).

Proof. We first prove that (6.6) is the unique compression for (6.5) assuming that $x_k^\star(C) = x_k^\star(S)$, for $k \in \{0, \dots, q_1+1\}$.

We start by showing that C is a compression for (6.5). Let $\bar{\delta}$ be any scenario in S, we need to show that $\bar{\delta} \in \mathcal{A}(C)$. Note that such a $\bar{\delta}$ either belongs to the discrete part of (6.6), or is feasible to the problem P_{q_1+1}. In the former case, $\bar{\delta}$ is in (6.6) by definition. In the latter case, it is also in $\mathcal{A}(C)$ since all these scenarios are in $\{\delta : g(x^\star(S), \delta) \leq 0\}$ (since $x^\star(S) = x^\star(C)$). This shows that (6.6) is a compression set for (6.5).

Before we proceed to the uniqueness proof, note that $|C| = r + d$ by Remark 6. With this in mind, let C', $C' \neq C$, be another compression of cardinality equal to $r + d$ for the mapping in (6.5). Let $\bar{k}$ be the minimum k for which $x^\star_k(S) = x^\star_k(C) \neq x^\star_k(C')$. Pick $\bar{\delta} \in \text{supp}(x^\star_{\bar{k}}(S)) \setminus \text{supp}(x^\star_{\bar{k}}(C'))$ such that $\bar{\delta} \in C \setminus C'$, such a $\bar{\delta}$ exists as otherwise we would contradict the fact that $x^\star_{\bar{k}}(S) \neq x^\star_{\bar{k}}(C')$. A similar argument has been used in the proof of Proposition 9, item *ii*), in Chapter 5, inspired by Lemma 2.12 of [22]. Hence, due to the fact that $\bar{\delta} \notin C \setminus C'$ we have that $\bar{\delta} \notin \text{supp}(x^\star_k(C'))$, for all $k \in \{0, \dots, q_1 + 1\}$, and in particular $\bar{\delta} \notin \text{supp}(x^\star(C'))$. Notice that $\bar{J} = \text{supp}(x^\star(C')) \subset C' \setminus \{\bigcup_{j=0}^{\bar{k}-1} \text{supp}(x^\star_j(S)) \cup \{\bar{\delta}\}\}$ since $x^\star_k(S) = x^\star_k(C')$ for all $k \in \{0, \dots, \bar{k} - 1\}$. As a consequence, by Assumption 14 this would imply that $g(z^\star(\bar{J}), \bar{\delta}) = g(x^\star(C'), \bar{\delta}) > 0$ (recall that $z^\star(\bar{J}) = x^\star(C')$), which contradicts the fact that $\bar{\delta} \in \mathcal{A}(C')$.

To conclude the proof, it remains to be shown that $x^\star_k(S) = x^\star_k(C)$ for any $k \in \{0, \dots, q_1 + 1\}$. This can be done by induction. For $k = 0$, note that $x^\star_0(S) = x^\star_0(C)$ since $\text{supp}(x^\star_0(S)) \subset C$. Suppose $x^\star_k(C) = x^\star_k(S)$ for all $k \in \{0, \dots, \bar{k}\}$, and consider $x^\star_{\bar{k}+1}(S)$. Since $R_{\bar{k}+1}(C) = R_{\bar{k}}(C) \cup \text{supp}(x^\star_{\bar{k}}(S))$ and $R_{\bar{k}}(C) = R_{\bar{k}}(S)$ and $\text{supp}(x^\star_{\bar{k}}(S)) = \text{supp}(x^\star_{\bar{k}}(C))$ by the induction hypothesis, we have that $\text{supp}(x^\star_{\bar{k}+1}(S)) \subset C \setminus R_{\bar{k}+1}(S)$, so $x^\star_{\bar{k}+1}(S) = x^\star_{\bar{k}+1}(C)$. This shows that $x^\star_k(S) = x^\star_k(C)$ for all $k \in \{0, \dots, q_1\}$. In the last stage, where only a subset of the support scenarios is discarded, we can use a similar argument. In fact, as consequence of the fact that $\text{supp}(x^\star_{q_1}(S)) = \text{supp}(x^\star_{q_1}(C))$ we have that $\bar{R}(S) = \bar{R}(C)$. However, this implies that $R_{q_1+1}(C) = R_{q_1+1}(S)$ and then $x^\star_{q_1+1}(C) = x^\star_{q_1+1}(S)$. This concludes the proof of the proposition. □

The next theorem follows from Proposition 11 and Theorem 1, Chapter 2.

Theorem 10. *Consider the removal scheme encoded by* (6.3) *and suppose Assumptions 13 and 14 hold. Let* $S = \{\delta_1, \ldots, \delta_m\}$ *be i.i.d. samples from the unknown distribution* $\mathbb{P}$, $r < m$ *be the number of discarded scenarios, and* $\epsilon \in (0, 1)$ *be given. Write* $r = q_1 d + q_2$ *and denote as* $x^\star(S)$ *as in* (6.4)*. Then we have that*

$$\mathbb{P}^m\{S \in \Delta^m : \mathbb{P}\{\delta \in \Delta : g(x^\star(S), \delta)\} > \epsilon\} = \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}. \tag{6.7}$$

Proof. The result for $q_2 = 0$ has been proved in Theorem 7, Chapter 5. If $q_2 \neq 0$, then we have by Proposition 11 that a unique compression set exists with cardinality $r + d$. The right-hand side in (6.7) follows then *mutatis mutandis* from the proof of Theorem 7, Chapter 5, with the only difference that the cardinality of the compression set is different. □

6.3.2 Arbitrary number of removed scenarios without Assumption 14

In the previous section we have extended the analysis of the removal algorithm proposed in Chapter 5 to a general number of discarded scenarios by relying on Assumption 14. In this section, we indicate possible extensions of such procedure without any further restriction on the underlying scenario program.

Indeed, the results in Section 6.3.1 rely on the fact that the cardinality of the set (6.6) is equal to $r + d$ (which is an immediate consequence of Proposition 10). However, this requires imposing Assumption 14 which effectively guarantees that realisations like the one Figure 6.1a occur with probability one. In this section we drop Assumption 14 and consider the more general case situation where realisations like the one of Figure 6.1b occur with non-zero probability. We provide a bound on the probability of constraint violation and show that for such cases there is no incentive in removing scenarios whose number is not an integer multiple of the dimension of the decision space.

In the general case where the realisation of Figure 6.1b happens with non-zero probability we cannot claim the bound of Theorem 10, as they can be no compression set of size equal to $r + d$. We can, however, establish a more conservative bound on the probability of constraint violation, as described in the sequel.

Theorem 11. *Suppose that Assumption 13 holds. Consider the removal scheme described in* (6.3). *Let* $S = \{\delta_1, \ldots, \delta_m\}$ *be i.i.d. samples from an unknown distribution* $\mathbb{P}$ *and* r *be an integer such that* $m > \lceil r \rceil_d + d$, *where* $\lceil r \rceil_d$ *is the smallest multiple of* d *that is larger than* r. *For any* $\epsilon \in (0, 1)$ *we have that*

$$\mathbb{P}^m\{S \in \Delta^m : \mathbb{P}\{\delta \in \Delta : g(x^\star(S), \delta) > 0\} > \epsilon\} \leq \sum_{i=0}^{\lceil r \rceil_d + d - 1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}, \quad (6.8)$$

Proof. See Section 6.5.1 . □

The proof of Theorem 11 follows closely the ones of Theorems 7 and 8 in Chapter 5 and the proof of Theorem 10 in this chapter, i.e., creating a specific mapping that involves the probability of constraint violation and showing that there exists a unique compression set of cardinality equal to $\lceil r \rceil_d + d$ associated to such a mapping. Implicitly, Theorem 11 states that tight results can only be achieved for general scenario programs if scenarios are removed in multiple of the dimension of the optimisation problem.

6.3.3 Min-max scenario programs

We can now consider the class of min-max scenario programs. Let $f : \mathcal{X} \times \Delta \to \mathbb{R}$ be a function, where $\mathcal{X}$ and Δ are defined as before. Assume $f(\cdot, \delta)$ is convex for all $\delta \in \Delta$. Given m samples $S = \{\delta_1, \ldots, \delta_m\}$, we want to solve the following min-max scenario program

$$\min_{x \in \mathcal{X}} \max_{\delta \in S} f(x, \delta), \quad (6.9)$$

which can be cast, through an epigraphic reformulation, as the following scenario program

$$\begin{aligned} \underset{(x,t) \in \mathcal{X} \times \mathbb{R}}{\text{minimise}} \quad & t \\ \text{subject to} \quad & f(x, \delta) \leq t, \text{ for all } \delta \in S. \end{aligned} \quad (6.10)$$

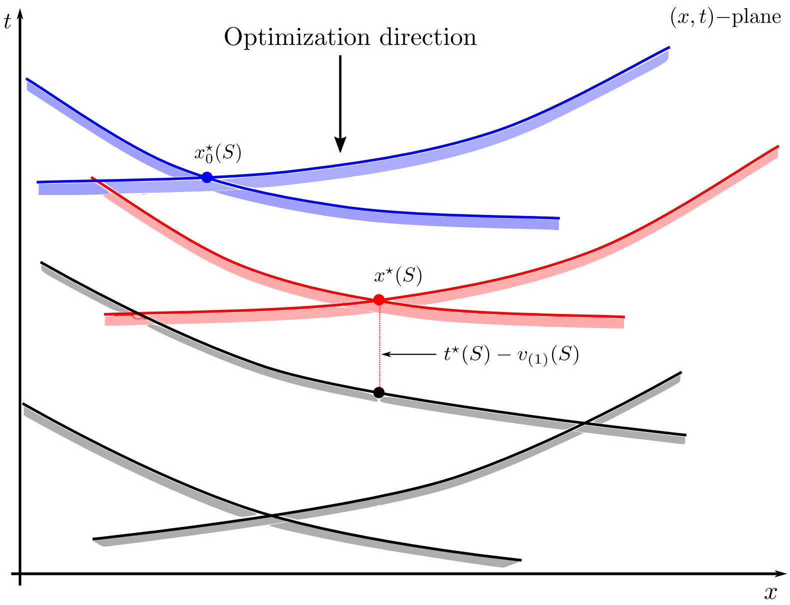


Figure 6.2: Alternative removal scheme suitable for min-max scenario programs with guaranteed bounds on the probability of constraint violation. We first remove scenarios by removing the support set, and then improve the cost at the last stage by moving downwards, if necessary. The blue and red scenarios correspond to first and second stages of the removal procedure. The dashed-red line defines $v_1(S)$.

Consider the following removal scheme, which is inspired by the results in [35] and [57] where the empirical cost for min-max scenario programs is characterised. For a given positive integer r such that $m > r + d + 1$ we proceed similarly as in the procedure described by (6.3), i.e., writing $r = q_1(d+1) + q_2$ and removing $q_1(d+1)$ scenarios by means of a cascade of scenarios programs in which the support set is removed at each stage. However, at the (q_1+1)-th stage, rather than choosing a subset of size q_2 from $\text{supp}((x^\star_{q_1}(S), t^\star_{q_1}(S)))$ to be discarded we compute the quantity

$$v_i(S) = t^\star_{q_1}(S) - f(x^\star_{q_1}(S), \delta_i), \text{ for all } \delta_i \in S \setminus \{R_{q_1}(S) \cup \text{supp}((x^\star_{q_1}(S), t^\star_{q_1}(S)))\}, \tag{6.11}$$

where $(x^\star_k(S), t^\star_k(S))$, $k \in \{0, \dots, q_1\}$, is the optimal solution of the scenario program (6.10), treated as a particular instance of the scenario program (6.3). It is important to notice that each $v_i(S)$ is related to the vertical distance between $t^\star_{q_1}(S)$ and the intersection of the constraint generated by the i-th scenario with the vertical line that passes through $x^\star_{q_1}(S)$ (see Figure 6.2 for an illustration). We then pick the q_2-th smallest $v_i(S)$ and denote them as $v_{(1)}(S) < v_{(2)}(S) < \ldots < v_{(q_2)}(S)$. The

q_2-th layer probability of constraint violation associated to the optimal solution of (6.10) is then given by (denoting $x^\star_{q_1}(S) = x^\star(S)$ and $t^\star_{q_1}(S) = t^\star(S)$)

$$V_{q_2}(S) = \mathbb{P}\{\delta \in \Delta : f(x^\star(S), \delta) > t^\star(S) - v_{(q_2)}(S)\}, \tag{6.12}$$

which constitutes the probability that an unseen sample has a cost greater than $t^\star(S) - v_{(q_2)}(S)$. An illustration of this procedure for $d = 1, r = 3$, and $m = 9$ is depicted in Figure 6.2. Under this setting, we can state the following theorem.

Theorem 12. *Consider the removal scheme described in this section and let* $V_{q_2}(S)$ *be defined as in* (6.12)*. Let* $S = \{\delta_1, \ldots, \delta_m\}$ *be i.i.d. samples from an unknown distribution* $\mathbb{P}$ *and* r *be an integer such that* $m > r + d$*. If the min-max scenario program* (6.9) *admits a unique solution, then for any* $\epsilon \in (0, 1)$ *we have that*

$$\mathbb{P}^m\{S \in \Delta^m : V_{q_2}(S) > \epsilon\} \leq \sum_{i=0}^{r+d-1} \binom{m}{i} \epsilon^i (1-\epsilon)^{m-i}.$$

Proof. See Section 6.5.2. □

6.4 Conclusion

We have analysed how the removal procedure proposed in Chapter 5 yields tight results on the probability of constraint violation when an arbitrary number of scenarios are discarded, as opposed to being a multiple of the dimension of the decision space.

We have shown that under Assumption 14 the constraint on the number of removed scenarios can be lifted while providing a feasibility bound that is based on a compression set of size $r + d$. However, more research is necessary to characterise how large is the class of scenario programs satisfying Assumption 14. To overcome such a shortcoming we proposed a more conservative bound that holds for all fully-supported scenario program. There are two facets to such a bound. On the one hand we show better feasibility guarantees for the resulting solution than the standard sampling-and-discarding bound applied to the case where an arbitrary number of scenarios is discarded. On the other hand we highlight an inherent property of the considered removal scheme, namely, the fact that removing a number of scenarios that is not an integer multiple of the dimension of the decision space introduces an additional conservatism in general.

Finally, we have also analysed the so-called min-max scenario programs by combining the removal strategy proposed in Chapter 5 with the one presented in [35] and [57].

6.5 Proofs of Chapter 6

6.5.1 Proof of Theorem 11

The proof of Theorem 11 is divided into two steps. We first study the probability of constraint violation associated to the optimal solution of a scenario program for which only a subset of its support scenarios is removed. Then we combine this analysis with the removal scheme of Chapter 5 to produce the bound of Theorem 11.

Step 1: Removing a subset of the support scenarios

Consider a cascade of two scenario programs as in (6.1) where one is obtained from the other by removing a subset of the support scenarios. Denote these scenario programs by SC_1 and SC_2, respectively, to distinguish them from the P_k in the removal procedure described in Section 6.2. Let SC_1 be

$$\begin{aligned} \mathrm{SC}_1 : \underset{x\in\mathcal{X}}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & g(x,\delta) \le 0, \text{ for all } \delta \in S. \end{aligned} \tag{6.13}$$

Denote by $v^\star(S)$ the optimal solution of (6.13) and denote, as before, by $\mathrm{supp}(v^\star(S))$ its support set. To define SC_2, fix any $0 < q_2 < d$, and let $M(S)$, with $|M(S)| = q_2$, be the subset of $\mathrm{supp}(v^\star(S))$ containing the q_2 smallest scenarios in $\mathrm{supp}(v^\star(S))$ according to the order $<_\sigma$. Then, let SC_2 be

$$\begin{aligned} \mathrm{SC}_2 : \underset{x\in\mathcal{X}}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & g(x,\delta) \le 0, \text{ for all } \delta \in S \setminus M(S). \end{aligned} \tag{6.14}$$

We denote the optimal solution of (6.14) by $w^\star(S)$ and its support set by $\text{supp}(w^\star(S))$. To analyse the probability of constraint violation properties associated with $w^\star(S)$, we first define for an arbitrary set of samples $C \subset S$ the set $N(C)$ containing the $|\text{supp}(v^\star(C)) \cap \text{supp}(w^\star(C))|$ smallest scenarios from $C \setminus \{\text{supp}(v^\star(C)) \cup \text{supp}(w^\star(C))\}$.

The reader may refer to Figure 6.1 for a motivation to the definitions of SC_1 and SC_2. In a comparison with the notation of Figure 6.1 we have that $v^\star(S) = x_0^\star(S)$ and $w^\star(S) = x^\star(S)$ (i.e., SC_1 plays the role of P_0 and SC_2 that of P_1); hence $|\text{supp}(v^\star(C)) \cap \text{supp}(w^\star(C))|$ is equal to the number of scenarios that belong to both support sets of SC_1 and SC_2, e.g., the scenarios are depicted in red in Figure 6.1. To encompass the fact that the realization in Figure 6.1b may happen with non-zero probability and to obtain a compression set with a cardinality that is uniform with respect to possible realizations, we need to append additional scenarios by forming the set $N(C)$ above. We believe that introducing SC_1, SC_2 as well as their related optimal solutions and support sets helps us to study the feasibility properties of a scenario program when only a subset of the support set is removed.

Similarly as in the proof of Theorem 7, Chapter 5, we establish a guarantee on the probability of constraint violation associated to $w^\star(S)$ by showing that there exists a compression scheme associated with such a removal procedure. To this end, we introduce the mapping $\mathcal{B} : \Delta^m \to 2^\Delta$

$$\mathcal{B}(C) = \{\mathcal{B}_1(C) \cap \mathcal{B}_2(C) \cap \mathcal{B}_3(C)\} \cup \bigcup_{\delta \in M(C) \cup N(C)} \delta, \tag{6.15}$$

with $\mathcal{B}_1(C) = \{\delta \in \Delta : g(v^\star(C), \delta) \leq 0\}$, $\mathcal{B}_2(C) = \{\delta \in \Delta : g(w^\star(C), \delta) \leq 0\}$, and

$$\mathcal{B}_3(C) = \left\{\delta \in \Delta : \delta \geq_\sigma \max_{\xi \in N(C)} \xi\right\} \cup \text{supp}(w^\star(C)).$$

Note that $\mathcal{B}_1(C) \cap \mathcal{B}_2(C)$ contains the scenarios that satisfy both of the interim solutions $v^\star(C)$ and $w^\star(C)$, while $\mathcal{B}_3(C)$ contains scenarios that are either larger than or equal to the maximum scenario[3] in $N(C)$ or that are in $\text{supp}(w^\star(S))$. In

[3]Formally, the ordering σ^{-1} is only defined on the finite set S. However, given any finite set S and under mild conditions on the uncertainty space Δ, one may extend σ^{-1} to the whole space Δ in a way that its restriction to S is the original bijection.

fact, the next proposition shows that

$$C = \operatorname{supp}(v^\star(S)) \cup \operatorname{supp}(w^\star(S)) \cup \bigcup_{\delta_j \in N(S)} \delta_j \tag{6.16}$$

is the unique compression of cardinality equal to $2d$ for (6.15).

Proposition 12. *Let $0 < q_2 < d$ be a given integer. Consider the cascade of two scenarios programs SC_1 and SC_2 as in (6.13) and (6.14), respectively. Suppose that the realization of Figure 6.1b happens with non-zero probability, i.e., suppose that, for all $m \in \mathbb{N}$, $\mathbb{P}^m\{S \in \Delta^m : |\operatorname{supp}(v^\star(S)) \cap \operatorname{supp}(w^\star(S))| = 0\} > 0$. Then, we have that:*

- *a) There exists a realization of scenarios S such that no compression of size smaller than $2d$ exists for the mapping $\mathcal{B}$ in (6.15).*
- *b) The set C in (6.16) is the unique compression set of cardinality $2d$ for the mapping $\mathcal{B}$ in (6.15).*

Remark 7. *Proposition 12 establishes compression properties related to a removal scheme that discards only a subset of the support scenarios of a scenario program, i.e., the set $M(C)$ above. A striking feature of this scheme is the fact that it may not yield tight bounds on the probability of constraint violation associated to $w^\star(C)$, as we may not have a compression set of cardinality equal to $d + q_2 < 2d$.*

Proof. Item a). We argue by contradiction. Let $S \subset \Delta$ be a set with cardinality m and assume that there exists a compression C' of cardinality $d' < 2d$ for the mapping $\mathcal{B}$ in (6.15). Fix a realization S that yields $N(S) = \emptyset$, i.e., one in which the support sets $\operatorname{supp}(v^\star(S))$ and $\operatorname{supp}(w^\star(S))$ are disjoint (e.g., see Figure 6.1b). Note that such a realization exists by assumption. As the cardinality of C' is strictly smaller than $2d$ we can find a scenario in $\{\operatorname{supp}(v^\star(S)) \cup \operatorname{supp}(w^\star(S))\} \setminus C'$, since the union of the support sets has cardinality equal to $2d$.

Let $\bar{\delta}$ be an element in $\{\operatorname{supp}(v^\star(S)) \cup \operatorname{supp}(w^\star(S))\} \setminus C'$. Such a $\bar{\delta}$ is either in $\operatorname{supp}(v^\star(S)) \setminus C'$ or in $\operatorname{supp}(w^\star(S)) \setminus C'$. Assume that $\bar{\delta} \in \operatorname{supp}(v^\star(S)) \setminus C'$, then the set $\mathrm{supp}(v^\star(S)) \setminus C'$ is non-empty. We next show that there exists a $\bar{\delta} \in \mathrm{supp}(v^\star(S)) \setminus C'$ such that $g(v^\star(C'), \bar{\delta}) > 0$. Recall that by the definition of a compression set we must have $g(v^\star(C'), \delta) \leq 0$ for all $\delta \in S$, so the existence of such a $\bar{\delta}$ implies that $\mathrm{supp}(v^\star(S))$ must be contained in C'. To this end, suppose for the sake of contradiction that $g(v^\star(C'), \bar{\delta}) \leq 0$ for all $\bar{\delta} \in \mathrm{supp}(v^\star(S)) \setminus C'$. This means that $v^\star(C')$ can be obtained by the following scenario program

$$\begin{aligned} \underset{x \in \mathbb{R}^d}{\text{minimise}} \quad & c^\top x \\ \text{subject to} \quad & g(x, \delta) \leq 0, \text{ for all } \delta \in C' \cup \mathrm{supp}(v^\star(S)), \end{aligned}$$

as adding the scenarios in $\mathrm{supp}(v^\star(S)) \setminus C'$ does not change the optimal cost. However, by the definition of support set and due to Assumption 13, this implies that $v^\star(C') = v^\star(S)$, which contradicts the fact that $\mathrm{supp}(v^\star(S)) \setminus C'$ is non-empty. Hence, we must have $g(v^\star(C'), \bar{\delta}) > 0$; however, this contradicts the fact that C' is a compression set for the mapping $\mathcal{B}$ in (6.15). In other words, if C' is a compression set of cardinality d then $\bar{\delta} \in \mathrm{supp}(w^\star(S)) \setminus C'$.

Since $\mathrm{supp}(v^\star(S)) \subset C'$, we must have that $v^\star(S) = v^\star(C')$ by Assumption 13, which then implies $M(S) = M(C')$. Changing S by $S \setminus \{\mathrm{supp}(v^\star(S)) \cup M(S)\}$ and C' by $C' \setminus \{\mathrm{supp}(v^\star(S)) \cup M(S)\}$ we can argue similarly as above to conclude that if $\mathrm{supp}(w^\star(S)) \setminus C'$ is not empty, then we can find an element in $\bar{\delta} \in \mathrm{supp}(w^\star(S)) \setminus C'$ such that $g(w^\star(C'), \bar{\delta}) > 0$, which contradicts the fact that C' is a compression. This concludes the proof of item a).

Item b). (Existence) We start the proof by showing that the set (6.16) is a compression for the mapping $\mathcal{B}$ in (6.15). To this end, we need to show that $\delta \in \mathcal{B}(C)$ for all $\delta \in S$. By the choice of C in (6.16) and under Assumption 1, we note that $v^\star(C) = v^\star(S)$ and $w^\star(C) = w^\star(S)$, which then implies $M(C) = M(S)$ and $N(C) = N(S)$. Pick $\bar{\delta} \in C$ and let us show that $\bar{\delta} \in \mathcal{B}(C)$. Suppose $\bar{\delta} \in \mathrm{supp}(v^\star(C))$. In this case we have two options: (1) either $\bar{\delta} \in M(S)$, which belongs to the discrete part of $\mathcal{B}(C)$; or (2) $\bar{\delta} \notin M(S)$, in which case it can be either in the support of $\mathrm{supp}(w^\star(S))$ or not. If $\bar{\delta} \in \mathrm{supp}(w^\star(S))$, then it belongs to $\mathcal{B}_1(C) \cap \mathcal{B}_2(C) \cap \mathcal{B}_3(C)$. The fact that such a $\bar{\delta}$ belongs to $\mathcal{B}_1(C) \cap \mathcal{B}_2(C)$ is clear due to $g(v^\star(S), \bar{\delta}) \leq 0$ and

$g(w^\star(S), \bar{\delta}) \leq 0$, while $\bar{\delta} \in \mathcal{B}_3(C)$ follows by definition, since $\mathrm{supp}(w^\star(S)) \subset \mathcal{B}_3(C)$. Otherwise, if $\bar{\delta} \in \mathrm{supp}(v^\star(S)) \setminus \mathrm{supp}(w^\star(S))$ then it either belongs to $N(S)$, which then implies that $\bar{\delta} \in \mathcal{B}(C)$, or $\bar{\delta} \in \mathrm{supp}(v^\star(S)) \setminus \{\mathrm{supp}(w^\star(S)) \cup N(S)\}$, hence it belongs to $\mathcal{B}_1(C) \cap \mathcal{B}_2(C)$ by definition, and to $\mathcal{B}_3(C)$ due to the fact that such a $\bar{\delta}$ must satisfy $\bar{\delta} \geq_\sigma \max_{\xi \in N(S)} \xi$. This shows that $\delta \in \mathcal{B}(C)$ for all $\delta \in \mathrm{supp}(v^\star(C))$.

Suppose now that $\bar{\delta} \in \mathrm{supp}(w^\star(C))$. It is straightforward to show that $\bar{\delta} \in \mathcal{B}_1(C) \cap \mathcal{B}_2(C) \cap \mathcal{B}_3(C)$ by means of similar arguments as above, so we have that $\bar{\delta} \in \mathcal{B}(C)$. Besides, if $\bar{\delta} \in N(C)$, then it belongs to the discrete part of $\mathcal{B}(C)$. Therefore, in any case if $\bar{\delta} \in C$, then $\bar{\delta} \in \mathcal{B}(C)$.

To conclude the existence proof, we need to show that if $\bar{\delta} \in S \setminus C$ then $\bar{\delta} \in \mathcal{B}(C)$. Since such a $\bar{\delta}$ is not in the discrete part of the mapping $\mathcal{B}(C)$, we need to show that $\bar{\delta} \in \mathcal{B}_1(C) \cap \mathcal{B}_2(C) \cap \mathcal{B}_3(C)$. As this $\bar{\delta}$ is feasible for both scenarios programs SC_1 and SC_2 we have that $\bar{\delta} \in \mathcal{B}_1(C) \cap \mathcal{B}_2(C)$. It remains to show that $\bar{\delta} \in \mathcal{B}_3(C)$. To this end, note that since $\bar{\delta} \notin C$ we have immediately that $\bar{\delta} >_\sigma \max_{\xi \in N(S)} \xi$, so it belongs to $\mathcal{B}_3(C)$. This shows that C given in (6.16) is a compression set for the mapping $\mathcal{B}$ in (6.15), thus concluding the existence part of the proof.

(Uniqueness) We divide the uniqueness proof into two cases: $N(S) = \emptyset$ and $N(S) \neq \emptyset$. In the former case, let C' be another compression set of size $2d$. Fix any $\bar{\delta} \in C \setminus C'$ and note that either $\bar{\delta} \in \mathrm{supp}(v^\star(C))$ or $\bar{\delta} \in \mathrm{supp}(w^\star(C))$ (note that $\bar{\delta}$ cannot belong to both sets due to the fact that $N(S) = N(C) = \emptyset$ is empty). If $\bar{\delta} \in \mathrm{supp}(v^\star(S))$ then a similar argument as in item a) (changing S by C in that argument) shows that there exists a $\bar{\delta} \in C \setminus C'$ such that $g(v^\star(C'), \bar{\delta}) > 0$, which contradicts the fact that C' is a compression. A similar argument also holds for $\bar{\delta} \in \mathrm{supp}(w^\star(C))$.

Consider now the case where $N(S) \neq \emptyset$. We proceed similarly as to the previous case and let C' be another compression of size $2d$. Fix any $\bar{\delta} \in C \setminus C'$ and note that $\bar{\delta}$ cannot belong to $\mathrm{supp}(v^\star(C)) \cup \mathrm{supp}(w^\star(C))$, as this would contradict, as before, the fact that C' is a compression. Hence, such a $\bar{\delta}$ must be an element of $N(C) \setminus C'$. Besides, since $\bar{\delta} \notin C'$ and C' is a compression, we must have that $\bar{\delta}$ is in

$\mathcal{B}_1(C') \cap \mathcal{B}_2(C') \cap \mathcal{B}_3(C')$. However, $\bar{\delta} \notin \mathcal{B}_3(C')$ as we have $\bar{\delta} <_\sigma \max_{\xi \in N(C')} \xi$, due to the fact that $C' \subset S$ and $\bar{\delta} \notin \text{supp}(w^\star(C')) \subset C'$, which imply that

$$\max_{\xi \in N(C')} \xi > \max_{\xi \in N(C) = N(S)} \xi,$$

This contradicts the fact that C' is a compression, thus concluding the proof of item b).

□

Step 2: Combining Proposition 12 with the results of Chapter 5

We are now in position to prove Theorem 11. Recall that d is the dimension of the optimization problem P_k and we are writing $r = q_1 d + q_2$, with $0 < q_2 < d$, where $m > \lceil r \rceil_d + d$. Define the mapping $\bar{\mathcal{A}} : \Delta^m \to 2^\Delta$ such that

$$\bar{\mathcal{A}}(C) = \mathcal{A}(C) \cap \{\mathcal{B}(C \setminus R_{q_1}(C)) \cup R_{q_1}(C)\}, \tag{6.17}$$

where $\mathcal{A}$ is the mapping given by

$$\mathcal{A}(C) = (\mathcal{A}_1(C) \cap \mathcal{A}_2(C)) \cup \mathcal{A}_3(C), \tag{6.18}$$

with, $\mathcal{A}_1(C) = \{\delta \in \Delta : g(x^\star_{q_1}(S), \delta) \leq 0\}$, $\mathcal{A}_3(C) = \bigcup_{k=0}^{q_1-1} \text{supp}(x^\star_k(C))$, and

$$\mathcal{A}_2(C) = \left\{ \bigcap_{k=0}^{q_1-1} \left\{ \delta \in \Delta : c^\top z^\star(J \cup \{\delta\}) \leq c^\top x^\star_k(S), \text{ for all } J \subset S \setminus R_k(S), \text{ with } |J| = d-1 \right\} \right\}.$$

The mapping $\mathcal{A}$ is associated with the removal procedure encoded by (6.3) when $q_2 = 0$ and has been introduced in Chapter 5 (see also [124], [125]), and $\mathcal{B}$ is the mapping of Proposition 12 with input given by $S \setminus R_{q_1}(S)$, rather than S. Note also that under this choice for the input of $\mathcal{B}$ we have $v^\star(S \setminus R_{q_1}(S)) = x^\star_{q_1}(S)$ and $w^\star(S \setminus R_{q_1}(S)) = x^\star_{q_1+1}(S) = x^\star(S)$ (see Section 6.2). In fact, under this notation, the scenario programs SC_1 and SC_2 in Proposition 12 correspond to P_{q_1} and P_{q_1+1}, respectively, in the description of Section 6.2.

We will show that the subset of the scenarios given by

$$C = \bigcup_{k=0}^{q_1} \text{supp}(x_k^\star(S)) \cup \text{supp}(x^\star(S)) \cup \bigcup_{j \in N(S)} \delta_j \tag{6.19}$$

is a compression set for the mapping $\bar{\mathcal{A}}$ in (6.17) – uniqueness will be shown in the sequel. First, note that such a C can be written as

$$C = C_1 \cup C_2,\ C_1 = \bigcup_{k=0}^{q_1} \text{supp}(x_k^\star(S)),$$
$$C_2 = \text{supp}(x_{q_1}^\star(S)) \cup \text{supp}(x^\star(S)) \cup \bigcup_{\delta \in N(S)} \delta.$$

The fact that C in (6.19) forms a compression set for the mapping $\bar{\mathcal{A}}$ follows trivially since C_1 and C_2 are compression sets for the removal procedure encoded by (6.3) due to 7 in Chapter 5 and Proposition 12, i.e., $\delta \in \mathcal{A}(C) \cap \{\mathcal{B}(C \setminus R_{q_1}(C)) \cup R_{q_1}(C)\}$ for all $\delta \in S$. Besides, observe that the cardinality of C is equal to $(q_1 + 2)d = \lceil q_1 d + q_2 \rceil_d + d = \lceil r \rceil_d + d$ due to definition of set $N(S)$ given in Proposition 12 and to the relation $r = q_1 d + q_2$.

We now show that the set C in (6.19) is the unique compression set of cardinality equal to $\lceil r \rceil_d + d$ for the mapping in (6.17). Suppose C' is another compression set of cardinality equal to $\lceil r \rceil_d + d$ for $\bar{\mathcal{A}}$. This means that $\delta \in \bar{\mathcal{A}}(C')$ for all $\delta \in S$. However, by the results in Chapter 5, we must have $C_1 \subset C'$; otherwise, there would exist another compression set of size $(q_1 + 1)d$ for the mapping $\mathcal{A}$. We also obtain that $\delta \in \mathcal{B}(C')$ for all $\delta \in S$. Since $C' \setminus R_{q_1}(S) \subset S \setminus R_{q_1}(S)$, by Proposition 12, we must also have that $C_2 \subset C$. However, as the cardinality of $C_1 \cup C_2$ is equal to $\lceil r \rceil_d + d$, this implies that $C' = C$, thus showing uniqueness of the compression set C in (6.19).

It remains to show how the existence and uniqueness of a compression set for the mapping $\bar{\mathcal{A}}$ can be used to produce the bound of Theorem 11. To this end, recall that (the dependence on C of the inner sets is omitted to simplify the notation)

$$\bar{\mathcal{A}}(C) = \underbrace{\{(\mathcal{A}_1 \cap \mathcal{A}_2) \cup \mathcal{A}_3\}}_{\mathcal{A}(C)} \cap \underbrace{\{(\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3) \cup \mathcal{B}_4\}}_{\mathcal{B}(C \setminus R_{q_1}(C)) \cup R_{q_1}(C)},$$

where we have defined $\mathcal{B}_4 = R_{q_1} \cup \bigcup_{\delta \in M \cup N} \delta$, which contains all the removed scenarios and potentially additional scenarios that compose the set $N(C)$ described in Proposition 12. After some elementary manipulations, we can prove that

$$\begin{aligned}\bar{\mathcal{A}}(C) &\subset (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3) \cup (\mathcal{A}_3 \cup \mathcal{B}_4) \\ &= (\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{B}_2 \cap \mathcal{B}_3) \cup (\mathcal{A}_3 \cup \mathcal{B}_4),\end{aligned} \tag{6.20}$$

where the second equality holds due to the fact that $x^\star_{q_1}(C) = v^\star(C \setminus R_{q_1}(C))$, which in turn implies that $\mathcal{A}_1(C) = \mathcal{B}_1(C \setminus R_{q_1}(C))$. Our ultimate goal is to bound the probability of $\mathcal{B}_2$. We can then use (6.20) to obtain the relation

$$\begin{aligned}&\mathbb{P}^m\{(\delta_1, \ldots, \delta_m) \in \Delta^m : \mathbb{P}\{\delta \notin \mathcal{B}_2(C \setminus R_{q_1}(C))\} > \epsilon\} \\ &\qquad \le \mathbb{P}^m\{(\delta_1, \ldots, \delta_m) \in \Delta^m : \mathbb{P}\{\delta \notin \bar{\mathcal{A}}(C)\} > \epsilon\}.\end{aligned}$$

However, note that the left-hand side of the above inequality is the probability of constraint violation we are interested in and the right-hand side can be upper bounded by Theorem 3 in [94] and the fact that there exists a unique compression set of size $\lceil r \rceil_d + d$ (as shown above), yielding the expression in the right-hand side of (6.8). This concludes the proof of Theorem 11.

6.5.2 Proof of Theorem 12

As for the case of Theorem 11, we divide the proof of Theorem 12 into two steps. The first step considers the removal procedure that improves the cost by moving downwards in the direction of the epigraphic variable when $0 < r < d+1$, i.e., $q_1 = 0$ and $q_2 = r$. The second step of the proof combines the feasibility guarantee obtained in the first step with Theorem 7 of Chapter 5.

Step 1: Moving downwards in the direction of the epigraphic variable

Assume that $q_1 = 0$ and $r = q_2 < d+1$. Consider the min-max scenario program

$$\begin{aligned}&\underset{(x,t) \in \mathcal{X} \times \mathbb{R}}{\text{minimise}} \quad t \\ &\text{subject to} \quad f(x, \delta) \le t, \ \text{for all } \delta \in S,\end{aligned} \tag{6.21}$$

where t is the epigraphic variable, and let $(x^\star(S), t^\star(S))$ be its optimal solution and $\mathrm{supp}((x^\star(S), t^\star(S)))$ be its support set. Define $v_i(S)$ and $v_{(i)}$ as described in (6.11), and recall that $v_{(i)}(S)$, $i = 1, \ldots, |S| - (d+1)$, is a monotonic sequence. We are interested in the quantity $V_{q_2}(S)$ defined in (6.12) with $(x^\star_{q_1}(S), t^\star_{q_1}(S)) = (x^\star(S), t^\star(S))$ (since $q_1 = 0$), which constitutes the probability that an unseen sample has a cost greater than $t^\star(S) - v_{(q_2)}(S)$.

To produce bounds on the tail distribution of $V_{q_2}(S)$ we consider the mapping[4] $\mathcal{B} : \Delta^m \to 2^\Delta$ as

$$\mathcal{B}(C) = \{\delta \in \Delta : f(x^\star(C), \delta) \leq t^\star(C) - v_{(r)}(C)\} \cup C, \tag{6.22}$$

which is the union of a discrete set containing the samples in C and the set of scenarios δ that generates a constraint that intersects the vertical line passing through $x^\star(S)$ below the value given by $t^\star(C) - v_{(r)}(C)$. Our strategy is to show that the set

$$C = \mathrm{supp}((x^\star(S), t^\star(S))) \cup \left\{ \bigcup_{j=1}^{r-1} \delta_{(j)} \right\}, \tag{6.23}$$

where $\delta_{(j)}$ denotes the scenario that leads to the j-th largest $v_j(S)$, i.e., $v_{(j)}(S) = t^\star(S) - f(x^\star(S), \delta_{(j)})$, is the unique compression set of cardinality equal to $d + r$ associated to the mapping $\mathcal{B}$ in (6.22). This is proved in the next proposition.

Proposition 13. *Given a set of samples $S = \{\delta_1, \ldots, \delta_m\}$. Consider the removal scheme encoded by* (6.21)*, and the mapping $\mathcal{B}$ as in* (6.22)*. Let $r = q_2$, with $0 < q_2 < d$ (i.e., $q_1 = 0$), then we have that the set C in* (6.23) *is the unique compression set associated to $\mathcal{B}$.*

Proof. (Existence) We need to show that $\delta \in \mathcal{B}(C)$ for all $\delta \in S$, with C given in (6.23). Note that if $\delta \in C$, then $\delta \in \mathcal{B}(C)$, as it belongs to the discrete part of the mapping $\mathcal{B}$ in (6.22). Then, it suffices to show that $\delta \in \mathcal{B}(C)$ for all $\delta \in S \setminus C$. To this end notice that any scenario $\delta_i \in S \setminus C$ leads to $v_{(r)}(C) \leq v_i(C)$, so

[4] Observe the overlapping notation. The mapping $\mathcal{B}$ in this section is not related to the mapping $\mathcal{B}$ in (6.15)

$f(x^\star(C), \delta_i) = t^\star(C) - v_i(C) \leq t^\star(C) - v_{(r)}(C)$. This shows that C in (6.6) is a compression set of cardinality equal to $r + d$ for the mapping $\mathcal{B}$ in (6.22).

(Uniqueness) To show uniqueness, assume that there exists another compression set C' of cardinality equal to $r+d$. Pick any $\bar{\delta} \in C \setminus C'$. We will show that if $C \setminus C'$ is non-empty we reach a contradiction with the fact that C' is a compression for the mapping $\mathcal{B}$ in (6.22), so we must have $C' = C$.

Suppose that $\bar{\delta} \in \text{supp}((x^\star(S), t^\star(S))) \setminus C'$. By Definition 15, Assumption 13 and the fact that the min-max scenario program admits a unique solution, we must have that $(x^\star(S), t^\star(S)) \neq (x^\star(C'), t^\star(C'))$ with[5] $t^\star(C') < t^\star(S)$. We claim that there exists a $\bar{\delta} \in \text{supp}(x^\star(C'), t^\star(C')) \setminus C'$ with the property that

$$t^\star(C') - f(x^\star(C'), \bar{\delta}) \leq 0. \tag{6.24}$$

Otherwise, if $t^\star(C') - f(x^\star(C'), \delta) > 0$ for all $\delta \in \text{supp}((x^\star(S), t^\star(S))) \setminus C'$, then $(x^\star(C'), t^\star(C'))$ would be feasible to the scenario program (6.21), which would then contradict optimality of $t^\star(S)$ since we know that $t^\star(C') < t^\star(S)$. Let any $\bar{\delta} \in \text{supp}((x^\star(S), t^\star(S))) \setminus C'$ that satisfies (6.24). We now show that it cannot be in $\mathcal{B}(C')$. To this end, suppose that $\bar{\delta} \in \mathcal{B}(C')$. Then, we have that

$$t^\star(C') \leq f(x^\star(C'), \bar{\delta}) \leq t^\star(C') - v_{(r)}(C'),$$

where the first inequality holds due to (6.24) and the second one to the fact that $\bar{\delta} \in \mathcal{B}(C')$. However, since $v_{(r)}(C') > 0$ by construction, we reach the contradiction $t^\star(C') < t^\star(C')$. This shows that if C' is a compression set, then $\text{supp}((x^\star(S), t^\star(S))) \subset C'$.

In other words, if C' is a compression and $\bar{\delta} \in C \setminus C'$, by the definition of C in (6.23) we must have that $\bar{\delta} \in \cup_{j=1}^{r-1} \delta_{(j)} \setminus C'$. This latter fact implies that $(x^\star(C), t^\star(C)) = (x^\star(C'), t^\star(C')) = (x^\star(S), t^\star(S))$. Moreover, since $C' \subset S$ and $v_{(r)}(S) = v_{(r)}(C)$, we must have that $v_{(r)}(C') > v_{(r)}(C)$. Let $\bar{\delta} \in C \setminus C'$ be such

[5] We cannot have $t^\star(C') = t^\star(S)$. In fact, if we had $t^\star(C') = t^\star(S)$, then by uniqueness of the min-max scenario program (6.9) we would have that $x^\star(S) = x^\star(C')$; however, this cannot happen since we know that $\bar{\delta}$ is in $\text{supp}((x^\star(S), t^\star(S)))$.

that $v_{(r)}(S) = t^\star(S) - f(x^\star(S), \bar{\delta})$ and for the sake of contradiction suppose that $\bar{\delta}$ belongs to $\mathcal{B}(C')$. We then obtain

$$v_{(r)}(C') \leq t^\star(S) - f(x^\star(S), \bar{\delta}) = v_{(r)}(S),$$

where the inequality follows from the assumption that $\bar{\delta} \in \mathcal{B}(C')$ and the fact that $(x^\star(C'), t^\star(C')) = (x^\star(S), t^\star(S))$, and the equality from the definition of $\bar{\delta}$. However, this contradicts the fact that $v_{(r)}(C') > v_{(r)}(S)$. Hence, we conclude that C in (6.23) is the unique compression set of cardinality equal to $(r+d)$ for the mapping $\mathcal{B}$ in (6.22). This concludes the proof of Proposition 13. □

Step 2: Combining Proposition 13 with Theorem 7, Chapter 5.

We treat the case of arbitrary r in a similar way as in the proof of Theorem 11. Indeed, we define $\bar{\mathcal{A}} : \Delta^m \to 2^\Delta$ as

$$\bar{\mathcal{A}}(C) = \mathcal{A}(C) \cap \{\mathcal{B}(C \setminus R_{q_1}(C)) \cup R_{q_1}(C)\}, \tag{6.25}$$

where the mapping $\mathcal{A}$ is given in (6.18) and $\mathcal{B}$ is given in (6.22). As before, the mapping $\mathcal{A}$ represents the first stage of the removal procedure where the support set associated to the scenario program (6.10) is discarded at each stage. The mapping $\mathcal{B}$ represents the last stage where we move downwards in the direction of the epigraphic variable.

Recall that, given $S = \{\delta_1, \ldots, \delta_m\}$, we write $r = q_1(d+1) + q_2$, with $0 < q_2 < d+1$, using the division algorithm. A compression candidate for the mapping $\bar{\mathcal{A}}$ is

$$C = \bigcup_{k=0}^{q_1} \mathrm{supp}((x_k^\star(S), t_k^\star(S))) \cup \bigcup_{j=1}^{q_2-1} \delta_{(j)}, \tag{6.26}$$

which has cardinality equal to $(d+1)(q_1+1) + q_2 - 1 = r + d$. Note that C in (6.26) can be written as

$$C = C_1 \cup C_2,$$

$$C_1 = \bigcup_{k=0}^{q_1} \mathrm{supp}((x_k^\star(S), t_k^\star(S))), \quad C_2 = \mathrm{supp}((x_{q_1}^\star(S), t_{q_1}^\star(S))) \cup \bigcup_{j=1}^{q_2-1} \delta_{(j)}.$$

To show that C is the unique compression set associated to the mapping $\bar{\mathcal{A}}$ in (6.25) we follow *mutatis mutandis* the corresponding arguments in the proof of Theorem 11, i.e., using the fact that the mappings $\mathcal{A}$ and $\mathcal{B}$ possess a unique compression set by Theorem 7 in Chapter 5 and Proposition 13, respectively.

Finally, the bound on the tail distribution of $V_{q_2}(S)$ can be obtained by means of elementary probability arguments on the mappings that compose $\bar{\mathcal{A}}$ in (6.25). Indeed, we obtain

$$\mathbb{P}^m\{S \in \Delta^m : V_{q_2}(S) > \epsilon\} \leq \mathbb{P}^m\{S \in \Delta^m : \mathbb{P}\{\delta \in \Delta : \delta \notin \bar{\mathcal{A}}(C)\} > \epsilon\}. \quad (6.27)$$

However, as there exists a unique compression set of cardinality equal to $r + d$ associated to $\bar{\mathcal{A}}(C)$, we can invoke Theorem 1, Chapter 2, to obtain the bound of Theorem 12, thus concluding the proof. □

7
Summary and future research directions

The role of optimisation problems is central to modern engineering applications. In this thesis we addressed three features that hinder the applicability of optimisation techniques, namely, scalability, the presence of integer decision variables, and the presence of uncertain constraints. Below we present a summary of the main results of this thesis, also discussing future research directions.

Chapter 3: Subgradient averaging for multi-agent optimisation

Scalability of optimisation problems was addressed in Chapter 3. Our approach focused on optimisation problems with separable objective functions and constraint sets and produced an algorithm that enables scalability by leveraging local computation.

We proposed a distributed algorithm that involves agents communicating over a network and sharing information with neighbouring agents only. These features allow for scalability when data is scattered and cannot be stored in a single processing unit. The proposed algorithmic scheme is based on subgradient averaging and converges to the optimal set of the centralised problem under time-varying communication networks, different constraint sets per agent and non-smooth objective functions. We have also characterised the rate at which the generated iterates converge to the optimal set, showing it recovers well-known rates for the centralised problem under similar assumptions.

There is several scope for improvements in the analysis of this algorithmic scheme. For instance, extensions of the results in Chapter 3 to encompass unbounded delays [17], [62], [86] would be important, as real communication networks are subject to delays that may undermine its convergence properties. The challenge here is to recognize the weakest assumption on the delays to reach a broad range of application, while still maintaining similar convergence properties as in Chapter 3.

Chapter 4: Distributed actuator selection

Solving optimisation programs in the presence of integer variables often leads to intractability, as there are no general polynomial-time algorithms for this class of problems. In Chapter 4, we have studied a combinatorial problem that consists in allocating a subset of actuators to maximise the trace of the controllability gramian of the resulting network. Chapter 4 shows that the feasible set of a relaxation of the problem is a polyhedron whose vertices have integer components, i.e., the matrix describing the feasible set is totally unimodular. Hence, the optimal solution of this formulation can be obtained through convex optimisation. We also leveraged this result and the structure of the problem to show how existing distributed algorithms can be employed to obtain such an optimal solution.

Interesting extensions include dealing with constraints in the control input and increasing the privacy level in the communication between agents. Besides, it is worth noticing that our approach relies on the fact that the optimal solution using the trace of the gramian as optimisation metric is on the vertices of a polyhedron. Providing guarantees to other more general metrics, such as those studied in [54], [68], [139], using convex relaxations is an interesting direction, as the only known approximations are based on submodularity properties and greedy strategies to select the actuators. Extending the analysis in [110] for time-varying networks to other more general metrics is another interesting direction.

Chapters 5 and 6: Scenario optimisation with discarded samples

Uncertain optimisation is – at least conceptually – the closest model to reality, as it can capture inaccuracies in the system's parameters and unmodelled dynamics. Unfortunately, such a model often leads to computational, as well as theoretical intractability. In this thesis, we have studied a randomised approximation of chance-constrained optimisation under the lens of the scenario approach theory, which relies on independent and identically distributed samples (or scenarios) to produce a feasible solution to the original, chance-constrained problem with high probability.

In Chapters 5 and 6, we studied the sampling-and-discarding approach to scenario optimisation, in which the decision maker is allowed to discard some of the scenarios, thus pursuing the typical trade-off between feasibility and performance. Chapter 5 analysed a specific removal scheme and proved an *a-priori* bound on the probability of constraint violation for the final solution that provides improvement compared to the state-of-the-art. We also show that the proposed bound is tight. The main limitation of the analysis of Chapter 5 is the fact that scenarios must be removed proportionally to the dimension of the optimisation problem.

In Chapter 6, we explore the extent to which the removal scheme of Chapter 5 can be applied to an arbitrary number of removed scenarios. We first show that the condition that has been employed in Chapter 5 to prove tightness of the proposed bound allows us to extend the considered removal scheme to an arbitrary number of removed scenarios. In case such a condition cannot be satisfied, we also propose a loose, but more general, bound on the probability of constraint violation. In fact, this latter result reveals a negative statement about the proposed removal scheme, i.e., it states that there is no advantage (in terms of the probability of constraint violation) to not remove scenarios in a number proportional to the dimension of the optimisation problem. We also explore the class of min-max scenario programs by combining the proposed removal scheme with an existing approach that improves the cost by moving downwards in the direction of the epigraphic variable. We show that better bounds on the probability of constraint violation can be obtained for this particular class of scenario programs.

This research direction still has a lot of scope to be explored. We are currently focusing on how this new, less conservative bound leads to better performance in power flow problems. Our intuition says that due to a tighter bound on the probability of constraint violation we are allowed to remove more constraints with respect to other removal strategies that rely on the bound in [30], while guaranteeing the same levels of confidence and probability of constraint violation. Hence, we expect to obtain better cost improvements using the proposed removal scheme. Another interesting research direction foresees to relax the requirement that all the scenarios are i.i.d., since some level of correlation among scenarios is expected in several applications, e.g., estimating the reachable sets [44].

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