Observational Robustness and Invariances in Reinforcement Learning via Lexicographic Objectives

Anonymous Author(s)
Affiliation
Address
`email`

Abstract

Policy robustness in Reinforcement Learning may not be desirable at any costs: the alterations caused by robustness requirements from otherwise optimal policies should be explainable, quantifiable and formally verifiable. In this work we study how policies can be *maximally robust* to arbitrary observational noise by analysing how they are altered by this noise through a stochastic linear operator interpretation of the disturbances, and establish connections between robustness and properties of the noise kernel and of the underlying MDPs. Then, we construct sufficient conditions for policy robustness, and propose a robustness-inducing scheme, applicable to any policy gradient algorithm, that formally trades off expected policy utility for robustness through *lexicographic optimisation*, while preserving convergence and sub-optimality of the original algorithm.

1 Introduction

Robustness in Reinforcement Learning (RL) [Morimoto and Doya, 2005] can be looked at from different perspectives: (1) distributional shifts in the training data with respect to the deployment stage Satia and Lave Jr [1973], Heger [1994], Nilim and El Ghaoui [2005], Xu and Mannor [2006]; (2) uncertainty in the model or observations [Pinto et al., 2017, Everett et al., 2021]; (3) adversarial attacks against actions [Pattanaik et al., 2017, Fischer et al., 2019]; and (4) sensitivity of neural networks (used as policy or value function approximators) towards input disturbances [Kos and Song, 2017, Huang et al., 2017]. Robustness does not naturally emerge in most RL settings, since agents are typically only trained in a single, unchanging environment: There is a trade-off between how robust a policy is and how close it is to the set of optimal policies in its training environment, and in safety-critical applications we may need to provide formal guarantees for this trade-off.

Motivation Consider a dynamical system where we need to synthesise a controller (policy) through a model-free approach. When using a simulator for training we expect the deployment of the controller in the real system to be affected by different sources of noise, possibly not predictable or modelled (*e.g.* for networked components we may have sensor faults, communication delays, *etc*). In safety-critical systems, robustness (in terms of successfully controlling the system under disturbances) should preserve formal guarantees, and plenty of effort has been put on developing formal convergence guarantees on policy gradient algorithms [Agarwal et al., 2021, Bhandari and Russo, 2019] which vanish when "robustifying" policies through regularisation or adversarial approaches. Therefore, for such applications one would need a scheme to regulate the robustness-utility trade-off in RL policies, that on the one hand preserves the formal guarantees of the original algorithms, and on the other attains sub-optimality conditions from the original problem. Additionally, if we do not know the structure of the disturbance (which holds in most applications), learning directly a policy for an arbitrarily disturbed environment will yield unexpected behaviours when deployed in the true system.

Lexicographic Reinforcement Learning (LRL) Recently, lexicographic optimisation [Isermann, 1982, Rentmeesters et al., 1996] has been applied to the multi-objective RL setting [Skalse et al., 2022b]. In an LRL setting with different reward-maximising objective functions $\{K_i\}_{1 \le i \le n}$, some objectives may be more important than others, and so we may want to obtain policies that solve the multi-objective problem in a lexicographically prioritised way, *i.e.*, "find the policies that optimise objective i (reasonably well), and from those the ones that optimise objective $i+1$ (reasonably well), and so on". There exist both value- and policy-based algorithms for LRL, and the approach is broadly applicable to (most) existing RL algorithms [Skalse et al., 2022b].

Previous Work In robustness against *model uncertainty*, the MDP may have noisy or uncertain reward signals or transition probabilities, as well as possible resulting *distributional shifts* in the training data [Heger, 1994, Xu and Mannor, 2006, Fu et al., 2018, Pattanaik et al., 2018, Pirotta et al., 2013, Abdullah et al., 2019], which connects to ideas on distributionally robust optimisation [Wiesemann et al., 2014, Van Parys et al., 2015]. One of the first examples is Heger [1994], where the author proposes using minimax approaches to learn Q functions that minimise the worst case total discounted cost in a general MDP setting. Derman et al. [2020] propose a Bayesian approach to deal with uncertainty in the transitions. Another robustness sub-problem is studied in the form of *adversarial attacks or disturbances* by considering adversarial attacks on policies or action selection in RL agents [Gleave et al., 2020, Lin et al., 2017, Tessler et al., 2019, Pan et al., 2019, Tan et al., 2020, Klima et al., 2019]. Recently, Gleave et al. [2020] propose the idea that instead of modifying observations, one could attack RL agents by swapping the policy for an adversarial one at given times. For a detailed review on Robust RL see Moos et al. [2022]. Our work focuses in the study of robustness versus *observational disturbances*, where agents observe a disturbed state measurement and use it as input for the policy [Kos and Song, 2017, Huang et al., 2017, Behzadan and Munir, 2017, Mandlekar et al., 2017, Zhang et al., 2020, 2021]. This problem emerges in many robotics applications, where one learns a policy through a simulator or human imitation, and then needs to rely on sensor data for a real-world deployment. In particular Mandlekar et al. [2017] consider both random and adversarial state perturbations, and introduce physically plausible generation of disturbances in the training of RL agents that make the resulting policy robust towards realistic disturbances. Zhang et al. [2020] propose a *state-adversarial* MDP framework, and utilise adversarial regularising terms that can be added to different deep RL algorithms to make the resulting policies more robust to observational disturbances, and Zhang et al. [2021] study how LSTM increases robustness with optimal state-perturbing adversaries.

1.1 Main Contributions

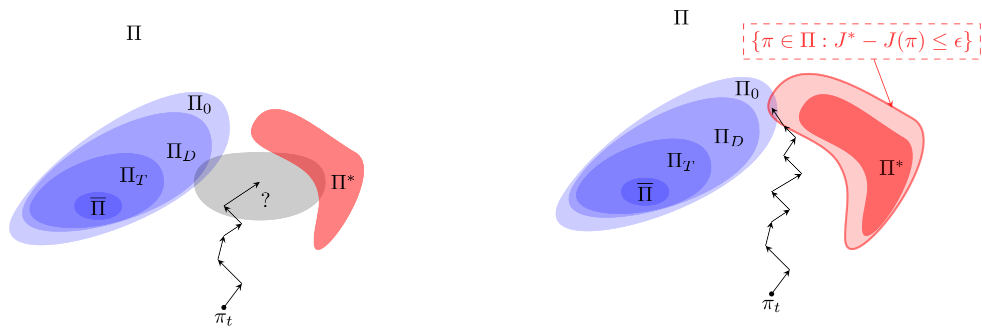
Most existing work on RL with observational disturbances proposes modifying RL algorithms (learning to deal with perturbations through linear combinations of regularising loss terms or adversarial terms) that come at the cost of *explainability* (in terms of sub-optimality bounds) and *verifiability*, since the induced changes in the new policies result in a loss of convergence guarantees. Our main contributions are summarised in the following points.

Structure of Robust Policy Sets. We consider general unknown stochastic disturbances and formulate a quantitative definition of observational robustness that allows us to characterise the sets of robust policies for any MDP in the form of operator-invariant sets. We analyse how the structure of these sets depends on the MDP and noise kernel, and obtain an inclusion relation (cf. the Inclusion Theorem, Section 3) providing intuition into how we can search for robust policies more effectively.[1]

Verifiable Robustness through LRL. The proposed characterisation and analysis allows us to cast robustness as a lexicographic optimisation objective and propose a meta-algorithm that can be applied to any existing policy gradient algorithm: Lexicographically Robust Policy Gradient (LRPG). Compared to existing approaches for observational robustness, LRPG allows us to:

1. Retain policy sub-optimality up to a specified tolerance while maximising robustness.
2. Formally control the utility-robustness trade-off through this design tolerance.

[1] We claim novelty on the application of such concepts to the understanding and improvement of robustness in disturbed observation RL. Although we have not found our results in previous work, there are strong connections between Sections 2-3 in this paper and the literature on planning for POMDPs [Spaan and Vlassis, 2004, Spaan, 2012] and MDP invariances [Ng et al., 1999, van der Pol et al., 2020, Skalse et al., 2022a].



(a) PG algorithms when robustness terms are added to the cost function *indiscriminately*.

(b) In LRPG, the policy is guaranteed (up to the original algorithm used) to converge to an ϵ ball of Π^*, and from those, the most robust ones.

Figure 1: Qualitative representation of the proposed LRPG algorithm, compared to usual robustness-inducing algorithms. The sets in blue are the maximally robust policies to be defined in the coming sections. Through LRPG we guarantee that the policies will only deviate a bounded distance from the original objective, and induce a search for robustness in the resulting valid policy set.

3. Preserve formal guarantees of the PG algorithm.

We provide numerical examples on how this approach is applied to existing policy gradient algorithms, comparing them to previous work and verifying how the previously mentioned Inclusion Theorem helps to induce more robust policies while retaining algorithm optimality. Figure 1 represents a qualitative interpretation of the results in this work (the structure of the robust sets will become clear in following sections).

1.2 Preliminaries

Notation We use calligraphic letters $\mathcal{A}$ for collections of sets and $\Delta(\mathcal{A})$ as the space of probability measures over $\mathcal{A}$. For two probability distributions P, P' defined on the same $\sigma-$algebra $\mathcal{F}$, $D_{TV}(P \| P') = \sup_{A \in \mathcal{F}} |P(A) - P'(A)|$ is the total variation distance. For two elements of a vector space we use $\langle \cdot, \cdot \rangle$ as the inner product. We use $\mathbf{1}_n$ as a column-vector of size n that has all entries equal to 1. We say that an MDP is *ergodic* if for any policy the resulting Markov Chain (MC) is ergodic. We say that S is a $n \times n$ row-stochastic matrix if $S_{ij} \geq 0$ and each row of S sums to 1.

Lexicographic Reinforcement Learning We provide an introduction to Policy-Based Lexicographic RL (PB-LRL) for an example with two objective functions. Consider a parameterised policy π_θ with $\theta \in \Theta$, and two objective functions K_1 and K_2. PB-LRL uses a multi-timescale optimisation scheme to optimise θ faster for higher-priority objectives, iteratively updating the constraints induced by these priorities and encoding them via Lagrangian relaxation techniques [Bertsekas, 1997]. Let $\theta' \in \operatorname{argmax}_\theta K_1(\theta)$. Then, PB-LRL can be used to find parameters $\theta'' = \operatorname{argmax}_\theta K_2(\theta)$, such that $K_1(\theta) \geq K_1(\theta') - \epsilon$. This is done through the estimated gradient ascent update:

$$\theta \leftarrow \operatorname{proj}_\Theta \left[\theta + \nabla_\theta \hat{K}(\theta)\right], \quad \lambda \leftarrow \operatorname{proj}_{\mathbb{R}_{\geq 0}} \left[\lambda + \eta_t(\hat{k}_1 - \epsilon_t - K_1(\theta))\right], \tag{1}$$

where $\hat{K}(\theta) := (\beta_t^1 + \lambda\beta_t^2) \cdot K_1(\theta) + \beta_t^2 \cdot K_2(\theta)$, λ is a Langrange multiplier, $\beta_t^1, \beta_t^2, \eta_t$ are learning rates[2], and $\hat{k}_1$ is an estimate of $K_1(\theta')$. Typically, we set $\epsilon_t \to 0$, though we can use other tolerances too, *e.g.*, $\epsilon_t = 0.9 \cdot \hat{k}_1$. For more details on the convergence proofs and technicalities of PB-LRL we refer the reader to Skalse et al. [2022b].

[2] We assume all learning rates in this work $\alpha_t(x, u) \in [0, 1]$ $(\beta_t, \eta_t ...)$ satisfy the conditions $\sum_{t=1}^{\infty} \alpha_t(x, u) = \infty$ and $\sum_{t=1}^{\infty} \alpha_t(x, u)^2 < \infty$.

2 Observationally Robust Reinforcement Learning

Robustness-inducing methods in model-free RL must address the following dilemma: How do we deal with uncertainty without an explicit mechanism to estimate such uncertainty during policy execution? Consider an example of an MDP where, at policy roll-out phase, there is a non-zero probability of measuring a "wrong" state. In such a scenario (even without adversarial uncertainty) optimal policies can be almost useless: measuring the wrong state can lead to executing unboundedly bad actions. This problem is represented by the following version of a noise-induced partially observable Markov Decision Process [Spaan, 2012].

Definition 2.1. An observationally-disturbed MDP (DOMDP) is (a POMDP) defined by the tuple (X, U, P, R, T, γ) where X is a finite set of states, U is a set of actions, $P : U \times X \mapsto \Delta(X)$ is a probability measure of the transitions between states and $R : X \times U \times X \mapsto \mathbb{R}$ is a reward function. The map $T : X \mapsto \Delta(X)$ is a stochastic kernel induced by some unknown noise signal, such that $T(y \mid x)$ is the probability of measuring y while the true state is x, and acts only on the state observations. At last $\gamma \in [0, 1]$ is a reward discount.

In a DOMDP[3] agents can measure the full state, but the measurement will be disturbed by some unknown random signal *in the policy deployment*. Unlike the POMDP setting, the agent has access to the true state x during learning of the policies (i.e., the simulator is noise-free), but has no information about the noise kernel T or a way to estimate it. The difficulty of acting in such DOMDP is that the transitions are actually undisturbed and a function of the true state x, but agents will have to act based on disturbed states $\tilde{x} \sim T(\cdot \mid x)$. We then need to construct policies that will be as robust as possible against such noise, without being able to construct noise estimates. This setting, which is distinguished from the POMDP one, reflects many robotic problems, where we can design a policy for ideal noise-less conditions, and we know that at deployment there will likely be noise, data corruption, adversarial perturbations, *etc.*, but we do not have a-priori knowledge on the structure of this disturbance. A (memoryless) policy for the agent is a stochastic kernel $\pi : X \mapsto \Delta(U)$. For simplicity, we overload notation on π, denoting by $\pi(x, u)$ as the probability of taking action u at state x under the stochastic policy π in the MDP, *i.e.*, $\pi(x, u) = \Pr\{u \mid x\}$. The value function of a policy π, $V^\pi : X \mapsto \mathbb{R}$, is given by $V^\pi(x_0) = \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t R(x_t, \pi(x_t), x_{t+1})]$. The action-value function of π (Q-function) is given by $Q^\pi(x, u) = \sum_{y \in X} P(x, u, y)(R(x, u, y) + \gamma V^\pi(y))$. We then define the objective function as $J(\pi) := \mathbb{E}_{x_0 \sim \mu_0}[V^\pi(x_0)]$ with μ_0 being a distribution of initial states, and we use $J^* := \max_\pi J(\pi)$ and π^* as the optimal policy. If a policy is parameterised by $\theta \in \Theta$ we write π_θ and $J(\theta)$.

Assumption 2.2. For any DOMDP and policy π, the resulting MC is irreducible and aperiodic.

We now formalise a notion of *observational robustness*. Firstly, due to the presence of the stochastic kernel T, the policy we are applying is altered as we are applying a collection of actions in a possibly wrong state. This behaviour can be formally captured by:

$$\Pr\{u \mid x, \pi, T\} = \langle \pi, T \rangle(x, u) := \sum_{y \in X} T(y \mid x)\pi(y, u), \tag{2}$$

where $\langle \pi, T \rangle : X \mapsto \Delta(U)$ is the *disturbed* policy, which averages the current policy given the error induced by the presence of the stochastic kernel. Notice that $\langle \cdot, T \rangle(x) : \Pi \mapsto \Delta(U)$ is an averaging operator yielding the alteration of the policy due to noise. We can then define the *robustness regret*[4]:

$$\rho(\pi, T) := J(\pi) - J(\langle \pi, T \rangle). \tag{3}$$

Definition 2.3 (Policy Robustness)**.** We say that a policy π is κ-*robust* against a stochastic kernel T if $\rho(\pi, T) \leq \kappa$. If π is 0-*robust* we say it is maximally robust. We define the sets of κ-robust policies, $\Pi_\kappa := \{\pi \in \Pi : \rho(\pi, T) \leq \kappa\}$, with Π_0 being the set of maximally robust policies.

One can motivate the characterisation and models above from a control perspective, where policies use as input discretised state measurements with possible sensor measurement errors. Formally ensuring robustness properties when learning RL policies will, in general, force the resulting policies to deviate from optimality in the undisturbed MDP. We propose then the following problem.

[3] Definition 2.1 is a generalised form of the State-Adversarial MDP used by Zhang et al. [2020]: the adversarial case is a particular form of DOMDP where T assigns probability 1 to one adversarial state.

[4] The robustness regret satisfies $\rho(\pi^*, T) \geq 0 \; \forall T$, and it allows us to directly compare the robustness regret with the utility regret of the policy.

Problem 1. *For a DOMDP and a given tolerance level ϵ, derive a policy π^ϵ that satisfies $J^* - J(\pi^\epsilon) \leq \epsilon$ as a prioritised objective and is as robust as possible according to Definition 2.3.*

3 Characterisation of Robust Policies

An important question to be addressed, before trying to synthesise robust policies through LRL, is what these robust policies look like, and how they are related to DOMDP properties. The robustness notion in Definition 2.3 is intuitive and it allows us to classify policies. We begin by exploring what are the types of policies that are maximally robust, starting with the set of constant policies and set of fix point of the operator $\langle \cdot, T \rangle$, whose formal descriptions are now provided.

Definition 3.1. A policy $\pi : X \mapsto \Delta(U)$ is said to be constant if $\pi(x) = \pi(y)$ for all $x, y \in X$, and the collection of all constant policies is denoted by $\overline{\Pi}$. A policy $\pi : X \mapsto \Delta(U)$ is called a fixed point of the operator $\langle \cdot, T \rangle$ if $\pi(x) = \langle \pi, T \rangle(x)$ for all $x \in X$. The collection of all fixed points will be denoted by Π_T.

In other words, a constant policy is any policy that yields the same action distribution for any state, and a fixed point policy is any policy whose action distributions are un-altered by the noise kernel. Observe furthermore that Π_T *only depends on the kernel* T and the set[5] X. We now present a proposition that links the two sets of policies in Definition 3.1 with our notion of robustness.

Proposition 3.2. *Consider a DOMDP as in Definition 2.1, the robustness notion given in Definition 2.3 and the concepts in Definition 3.1, then we have that $\overline{\Pi} \subseteq \Pi_T \subseteq \Pi_0$.*

The importance of Proposition 3.2 is that it allows us to produce (approximately) maximally robust policies by computing the distance of a policy to either the set of constant policies or to the fix point of the operator $\langle \cdot, T \rangle$, and this is at the core of the construction in Section 4. However, before this, let us introduce another set that is sandwiched between Π_0 and Π_T. Let us assume we have a policy iteration algorithm that employs an action-value function Q^π and policy π. The advantage function for π is defined as $A^\pi(x, u) := Q^\pi(x, u) - V^\pi(x)$ and can be used as a maximisation objective to learn optimal policies (as in, *e.g.*, A2C [Sutton et al., 1999], A3C [Mnih et al., 2016]). We can similarly define the *noise disadvantage* (a form of negative advantage) of policy π as:

$$D^\pi(x, T) := V^\pi(x) - \mathbb{E}_{u \sim \langle \pi, T \rangle (x)}[Q^\pi(x, u)], \tag{4}$$

which measures the difference of applying at state x an action according to the policy π with that of playing an action according to $\langle \pi, T \rangle$ and then continuing playing an action according to π. Our intuition says that if it happens to be the case that $D^\pi(x, T) = 0$ for all states in the DOMDP, then such a policy is maximally robust. And this is indeed the case, as shown in the next proposition.

Proposition 3.3. *Consider a DOMDP as in Definition 2.1 and the robustness notion as in Definition 2.3. If a policy π is such that $D^\pi(x, T) = 0$ for all $x \in X$, then π is maximally robust, i.e., let*

$$\Pi_D := \{\pi \in \Pi : \mu_\pi(x) D^\pi(x, T) = 0 \,\forall\, x \in X\},$$

then we have that $\Pi_D \subseteq \Pi_0$.

So far we have shown that both the set of fixed points $\overline{\Pi}$ and the set of policies for which the disadvantage function is equal to zero Π_D are contained in the set of maximally robust policies. More interesting is the fact that the inclusion established in Proposition 3.2 and the one in Proposition 3.3 can be linked in a natural way through the following Inclusion Theorem.

Theorem 3.4 (Inclusion Theorem)**.** *For a DOMDP with noise kernel T, consider the sets $\overline{\Pi}, \Pi_T, \Pi_D$ and Π_0. Then, the following inclusion relation holds:*

$$\overline{\Pi} \subseteq \Pi_T \subseteq \Pi_D \subseteq \Pi_0.$$

Additionally, the sets $\overline{\Pi}, \Pi_T$ are convex *for all MDPs and kernels T, but Π_D, Π_0 may not be.*

Let us reflect on the inclusion relations of Theorem 3.4. The inclusions are in general not strict, and in fact the geometry of the sets (as well as whether some of the relations are in fact equalities) is highly

[5]There is a (natural) bijection between the set of constant policies and the space $\Delta(U)$. The set of fixed points of the operator $\langle \cdot, T \rangle$ also has an algebraic characterisation in terms of the null space of the operator $\mathrm{Id}(\cdot) - \langle \cdot, T \rangle$. We are not exploiting the later characterisation in this paper.

dependent on the reward function, and in particular on the complexity (from an information-theoretic perspective) of the reward function. As an intuition, less complex reward functions (more uniform) will make the inclusions above expand to the entire policy set, and more complex reward functions will make the relations collapse to equalities. The following Corollary illustrates this.

Corollary 3.5. *For any* ergodic *DOMDP there exist reward functions $\overline{R}$ and $\underline{R}$ such that the resulting DOMDP satisfies: (i) $\Pi_D = \Pi_0 = \Pi$ (any policy is max. robust) if $R = \overline{\overline{R}}$, (ii) $\Pi_T = \Pi_D = \Pi_0$ (only fixed point policies are maximally robust) if $R = \underline{R}$.*

We can now summarise the insights from Theorem B.3 and Corollary 3.5 in the following conclusions: (1) The set $\overline{\Pi}$ is maximally robust, convex and *independent of the DOMDP*, (2) The set Π_T is maximally robust, convex, includes $\overline{\Pi}$, and its properties *only depend* on T, (3) The set Π_D includes Π_T and is maximally robust, but its properties *depend on the DOMDP*.

4 Robustness through Lexicographic Objectives

We have now characterised robustness in a DOMDP and explored the relation between the sets of policies that are robust according to the definition proposed. We have seen in the Inclusion Theorem that several classes of policies are maximally robust, and our goal now is to connect these results with lexicographic optimisation. To be able to apply LRL results to our robustness problem we need to first cast robustness as a valid objective to be maximised, and then show that a stochastic gradient descent approach would indeed find a global maximum of the objective, therefore yielding a maximally robust policy. Then, this robustness objective can be combined with a primary reward-maximising objective $K_1(\theta) = \mathbb{E}_{x_0 \sim \mu_0}[V^{\pi_\theta}(x_0)]$ and any algorithm with certified convergence to solve Problem 1. Policy-based LRL (PB-LRL) allows us to encode the idea that, when learning how to solve an RL task, robustness is important but *not at any price*, *i.e.*, we would like to solve the original objective reasonably well[6], and from those policies efficiently find the most robust one.

4.1 Robustness Objectives

We propose now a valid lexicographic objective for which a minimising solution yields a maximally robust policy. For this, we will perturb the policy during training according to the following logic. In the introduction, we emphasised that the motivation for this work comes partially from the fact that we may not know T in reality, or have a way to estimate it. However, the theoretical results until now depend on T. Our proposed solution to this lies in the results of Theorem 3.4. We can use a *design* generator $\tilde{T}$ to perturb the policy during training such that $\tilde{T}$ has the smallest possible fixed point set (i.e. the constant policy set), and any algorithm that drives the policy towards the set of fixed points of $\tilde{T}$ *will also drive the policy towards fixed points of* T: from Theorem 3.4, $\Pi_{\tilde{T}} \subseteq \Pi_T$.

Assumption 4.1. The design kernel $\tilde{T}$ satisfies $\Pi_{\tilde{T}} = \overline{\Pi}$

We discuss further the choice and implications of using a design kernel $\tilde{T}$ in Section 5. One of the messages of the Inclusion Theorem is the fact that fixed point policies are maximally robust. Consider the objective to be minimised:

$$K_{\tilde{T}}(\theta) = \sum_{x \in X} \mu_{\pi_\theta}(x) \frac{1}{2} \|\pi_\theta(x) - \langle \pi_\theta, \tilde{T} \rangle (x)\|_2^2, \tag{5}$$

Notice that optimising (5) projects the current policy onto the set of fixed points of the operator $\langle \cdot, \tilde{T} \rangle$, and due to Assumption 2.2, which requires $\mu_{\pi_\theta}(x) > 0$ for all $x \in X$, the optimal solution is equal to zero if and only if there exists a value of the parameter θ for which the corresponding π_θ is a fixed point of $\langle \cdot, \tilde{T} \rangle$. In practice, the objectives are computed for a batch of trajectory sampled states $X_s \subset X$, and averaged over $\frac{1}{|X_s|}$; we denote these approximations with a hat. By applying standard stochastic approximation arguments, we can prove that convergence is guaranteed for a SGD iteration using $\nabla_\theta \hat{K}_{\tilde{T}}(\theta)(x) = (\pi_\theta(x) - \pi_\theta(y)) \nabla_\theta \pi_\theta(x)$, $y \sim \tilde{T}(\cdot \mid x)$ to the optimal solution of problem 5. For details and a proof, see Lemma B.3 in Appendix B.

[6]The advantage of using LRL is that we need not know in advance how to define "reasonably well" for each new task. Additionally, we obtain a hyper-parameter that directly controls the trade-off between *robustness and optimality*: the tolerance ϵ. Through ϵ we determine how far we allow our resulting policy to be from an optimal policy in favour of it being more robust.

4.2 Lexicographically Robust Policy Gradient

We present now the proposed LRPG meta-algorithm to achieve lexicographic robustness for any policy gradient algorithm at choice. From Skalse et al. [2022b], the convergence of PB-LRL algorithms is guaranteed as long as the original policy gradient algorithm (such as PPO [Liu et al., 2019] or A2C [Konda and Tsitsiklis, 2000, Bhatnagar et al., 2009]) for each single objective converges. We can then combine Lemma B.3 with these results to guarantee that Lexicographically Robust Policy Gradient (LRPG), Algorithm 1, converges to a policy that maximise robustness while remaining (approximately) optimal with respect to R.

Algorithm 1 LRPG

input Simulator, $\tilde{T}$, ϵ
initialise θ, critic (if using), λ, $\{\beta_t^1, \beta_t^2, \eta\}$
set $t = 0$, $x_t \sim \mu_0$
while $t < \text{max\_iterations}$ **do**
  perform $u_t \sim \pi_\theta(x_t)$
  observe r_t, x_{t+1}, sample $y \sim \tilde{T}(\cdot \mid x)$
  if $\hat{K}_1(\theta)$ not converged **then**
    $\hat{k}_1 \leftarrow \hat{K}_1(\theta)$
  end if
  update critic (if using)
  update θ and λ using (1)
end while
output θ

Theorem 4.2. *Consider a DOMDP as in Definition 2.1 and let π_θ be a parameterised policy. Take $K_1(\theta) = \mathbb{E}_{x_0 \sim \mu_0}[V^{\pi_\theta}(x_0)]$ to be computed through a chosen algorithm (*e.g.*, A2C, PPO) that optimises $K_1(\theta)$, and let $K_2(\theta) = -K_{\tilde{T}}(\theta)$. Given an $\epsilon > 0$, if the iteration $\theta \leftarrow \mathrm{proj}_\Theta\left[\theta + \nabla_\theta \hat{K}_1\right]$ is guaranteed to converge to a parameter set θ^* that maximises K_1, and hence J (locally or globally), then LRPG converges* a.s. *under PB-LRL conditions to parameters θ^ϵ that satisfy:*

$$\theta^\epsilon \in \operatorname*{argmin}_{\theta \in \Theta'} K_{\tilde{T}}(\theta), \quad \textit{such that} \quad K_1^* \geq K_1(\theta^\epsilon) - \epsilon, \tag{6}$$

where $\Theta' = \Theta$ if θ^* is globally optimal and a compact local neighbourhood of θ^* otherwise.

We reflect again on Figure 1. The main idea behind LRPG is that by formally expanding the set of acceptable policies with respect to K_1, we may find robust policies more effectively while guaranteeing a minimum performance in terms of expected rewards. This addresses directly the premise behind Problem 1. In LRPG the first objective is still to minimise the distance $J^* - J(\pi)$ up to some tolerance. Then, from the policies that satisfy this constraint, we want to steer the learning algorithm towards a maximally robust policy, and we can do so without knowing T.

5 Considerations on Noise Generators

A natural question following Section 4.1 and the theoretical results in Section 4 is how to choose $\tilde{T}$, and how the choice influences the resulting policy robustness towards any other true T. In general, for any arbitrary policy utility landscape in a given MDP, there is no way of bounding the distance of the resulting policies for two different noise kernels T_1, T_2. As a counter-example, consider an MDP where there are 2 possible optimal policies π_1^*, π_2^*, and take these two policies to be maximally different, *i.e.* $D_{TV}(\pi_1^* \| \pi_2^*) = 1 \; \forall x \in X$. Then, when using LRPG to obtain a robust policy, a slight deviation in the choice of $\tilde{T}$ can cause the gradient descent scheme to deviate from converging to π_1^* to converging to π_2^*, yielding in principle a completely different policy. However, *the optimality of the policy* remains bounded: Through LRPG guarantees we know that, for both cases, the utility of the resulting policy will be at most ϵ far from the optimal. We can, thus, state the following.

Corollary 5.1. *Take T to be any arbitrary noise kernel, and $\tilde{T}$ to satisfy Assumption 4.1. Let π be a policy resulting from a LRPG algorithm. Assume that $\min_{\pi' \in \Pi_{\tilde{T}}} D_{TV}(\pi \| \pi') = a$ for some $a < 1$. Then, it holds for any T that $\min_{\pi' \in \Pi_T} D_{TV}(\pi \| \pi') \leq a$.*

That is, when using LRPG to obtain a robust policy π, the resulting policy is at most a far from the set of fixed points (and therefore a maximally robust policy) with respect to the true T. This is the key argument behind our choices for $\tilde{T}$: A priori, the most sensible choice is a kernel that has no other fixed point than the set of constant policies. This fixed point condition is satisfied in the discrete state case for any $\tilde{T}$ that induces an irreducible Markov Chain, and in continuous state for

	PPO on MiniGrid Environments				*A2C on MiniGrid Environments*			
Noise	Vanilla	$\text{LR}_{\text{PPO}}(K_T^u)$	$\text{LR}_{\text{PPO}}(K_T^g)$	SA-PPO	Vanilla	$\text{LR}_{\text{A2C}}(K_T^u)$	$\text{LR}_{\text{A2C}}(K_T^g)$	$\text{LR}_{\text{A2C}}(K_D)$
LavaGap								
$\emptyset$	**0.95±0.003**	**0.95±0.075**	**0.95±0.101**	0.94±0.068	**0.94±0.004**	**0.94±0.005**	**0.94±0.003**	**0.94±0.006**
T_1	0.80±0.041	**0.95±0.078**	0.93±0.124	0.88±0.064	0.83±0.061	**0.93±0.019**	0.89±0.032	0.91±0.088
T_2	0.92±0.015	**0.95±0.052**	**0.95±0.094**	0.93±0.050	0.89±0.029	**0.94±0.008**	0.93±0.011	0.93±0.021
LavaCrossing								
$\emptyset$	**0.95±0.023**	0.93±0.050	0.93±0.018	0.88±0.091	0.91±0.024	0.91±0.063	0.90±0.017	**0.92±0.034**
T_1	0.50±0.110	**0.92±0.053**	0.89±0.029	0.64±0.109	0.66±0.071	**0.78±0.111**	0.72±0.073	0.76±0.098
T_2	0.84±0.061	**0.92±0.050**	**0.92±0.021**	0.85±0.094	0.78±0.054	0.83±0.105	0.86±0.029	**0.87±0.063**
DynamicObstacles								
$\emptyset$	**0.91±0.002**	**0.91±0.008**	**0.91±0.007**	**0.91±0.131**	**0.91±0.011**	0.88±0.020	0.89±0.009	**0.91±0.013**
T_1	0.23±0.201	**0.77±0.102**	0.61±0.119	0.45±0.188	0.27±0.104	0.43±0.108	0.45±0.162	**0.56±0.270**
T_2	0.50±0.117	**0.75±0.075**	0.70±0.072	0.68±0.490	0.45±0.086	0.53±0.109	0.52±0.161	**0.67±0.203**

Table 1: Reward values gained by LRPG and baselines on discrete control tasks.

any $\tilde{T}$ that satisfies a reachability condition (*i.e.* for any $x_0 \in X$, there exists a finite time for which the probability of reaching any ball $B \subset X$ of radius $r > 0$ through a sequence $x_{t+1} = T(x_t)$ is measurable). This holds for (additive) uniform or Gaussian disturbances.

6 Experiments

We verify the theoretical results of LRPG in a series of experiments on discrete state/action safety-related environments [Chevalier-Boisvert et al., 2018], and in continuous control tasks. We use A2C [Sutton and Barto, 2018] (LR-A2C), PPO [Schulman et al., 2017] (LR-PPO) and SAC [Haarnoja et al., 2018] (LR-SAC) for our implementations of LRPG. In all cases, the lexicographic tolerance was set to $\epsilon = 0.99\hat{k}_1$ to deviate as little as possible from the primary objective. We compare against the baseline algorithms and against SA-PPO [Zhang et al., 2021] which is among the most effective (adversarial) robust RL approaches in literature. We trained 10 independent agents for each algorithm, and reported the scores of the median agent (as done in Zhang et al. [2020]) for 50 averaged roll-outs.

Sampling $\tilde{\mathbf{T}}$. To simulate $\tilde{T}$ we disturb x as $\tilde{x} = x + \xi$ for (1) a uniform bounded noise signal $\xi \sim \mathcal{U}_{[-b,b]}$ ($\tilde{T}^u$) and (2) and a Gaussian noise ($\tilde{T}^g$) such that $\xi \sim \mathcal{N}(0, 0.5)$. We test the resulting policies against a noiseless environment ($\emptyset$), a kernel $T_1 = \tilde{T}^u$, a kernel $T_2 = \tilde{T}^g$ and against two different state-adversarial noise configurations as proposed by Zhang et al. [2021] to evaluate how effective LRPG is at rejecting adversarial disturbances. See Appendix C for details and bounds used.

Robustness Objectives. If we do not have an estimator for the critic Q^π (*e.g.* PPO, A2C), Proposition 3.2 suggests that minimising the distance between π and $\langle \pi, T \rangle$ can serve as a proxy to minimise the robustness regret, so we use objectives as defined in (5). We aim to test the hypothesis introduced through this work: If we have an estimator for the critic Q^π we can obtain robustness without inducing regularity in the policy using D^π, yielding a larger policy subspace to steer towards, and hopefully achieving policies closer to optimal. With the goal of diving deeper into the results of Theorem 3.4, we consider the objective $K_D(\theta) := \sum_{x \in X} \mu_{\pi_\theta}(x) \frac{1}{2} \| D^{\pi_\theta}(x, T) \|_2^2$. We use both in our experimental results, by modifying A2C to retain a Q critic.

Robustness Results: Discrete Control. Firstly, we investigate the impact of LRPG PPO and A2C for discrete action-space problems on Gymnasium [Brockman et al., 2016]. *Minigrid-LavaGap* (fully observable), *Minigrid-LavaCrossing* (partially observable) are safe exploration tasks where the agent needs to navigate an environment with cliff-like regions. *Minigrid-DynamicObstacles* (stochastic, partially obserbvable) is a dynamic obstacle-avoidance environment. We use A2C to test the influence of K_D vs. K_T since the structure of the original cost functions are simpler than in PPO, and hence easier to compare between the scenarios above. With each objective function resulting in gradient descent steps that pull the policy towards different maximally robust sets ($K_T \to \Pi_T$ and $K_D \to \Pi_D$ respectively), we would expect to obtain increasing robustness for K_D. The results are presented in Table 1. See Appendix C for the results against adversarial noise, learning curves and detailed results.

	PPO on Continuous Environments				*SAC on Continuous Environments*		
Noise	Vanilla	LR_{PPO} (K_T^u)	LR_{PPO} (K_T^g)	SA-PPO	Vanilla	LR_{SAC} (K_T^u)	LR_{SAC} (K_T^g)
MountainCar							
$\emptyset$	**94.77±0.26**	93.17±0.89	94.66±1.61	88.69±3.93	93.52±0.05	**94.43±0.19**	93.84±0.05
T_1	88.67±1.41	91.46±1.22	**94.91±1.35**	88.41±3.99	1.89±65.31	71.81±13.04	**76.90±7.11**
T_2	92.22±1.11	92.40±1.28	**94.76±1.42**	89.32±3.79	-27.82±73.10	**72.93±8.57**	69.41±13.03
LunarLander							
$\emptyset$	267.99±38.04	**269.76±22.93**	243.08±37.03	220.18±98.78	268.96±51.52	275.17±14.04	**282.24±15.95**
T_1	156.09±22.87	**280.91±20.34**	182.80±49.26	164.53 ±45.48	128.18±17.73	**187.64±76.30**	153.81±33.16
T_2	158.02±46.57	**276.76±16.20**	212.62±37.56	221.84±73.61	140.92±20.61	**187.82±25.27**	158.18±28.60
BipedalWalker							
$\emptyset$	265.39±82.36	261.39±83.19	**276.66±44.85**	251.60±103.08	236.39 ±157.03	302.56±70.79	**313.56±52.17**
T_1	174.15±170.30	253.56±72.66	220.28±118.61	**264.69±61.63**	203.93 ±167.83	241.45±124.54	**241.60±139.93**
T_2	135.16±182.30	243.27±89.86	**265.37±80.60**	255.21±90.61	84.10 ±198.12	198.20±151.64	**229.75±166.87**

Table 2: Reward values gained by LRPG and baselines on continuous control tasks.

Robustness Results: Continuous Control. We studied the effectiveness of LRPG on continuous control problems, and compared LR-PPO and LR-SAC to baselines for three different continuous control environments on Gymnasium [Brockman et al., 2016]: *MountainCarContinuous*, *LunarLanderContinuous* and *BipedalWalker-v3*. Again, we trained 10 independent agents, and reported the scores of the median agent. The results for the different noise kernels tested are presented in Table 5.

7 Discussion

Experiments. We applied LRPG on PPO, A2C and SAC algorithms, for a set of discrete and continuous control environments. These environments are particularly sensitive to robustness problems; the rewards are sparse, and applying a sub-optimal action at any step of the trajectory often leads to terminal states with zero (or negative) reward. LRPG successfully induces lower robustness regrets in the tested scenarios, and the use of K_D as an objective (even though we did not prove the convergence of a gradient based method with such objective) yields a better compromise between robustness and rewards. When compared to recent observational robustness methods, LRPG obtains similar robustness results while *preserving the original guarantees of the chosen algorithm*[7]. However, the improvements seem to be smaller for SAC, possibly due to the different nature of policy losses (SAC uses a Q function as a loss).

Shortcomings. The motivation for LRPG comes from situations where, when deploying a model-free controller in a dynamical system, we do not have a way of estimating the noise generation. There is an alternative approach for robust RL, exploited in the reviewed literature, which consists in assuming a disturbance structure (*e.g.* adversarial noise) and training directly to optimise the rewards in the disturbed MDP. There is no clear answer on what approach is more rational, or more effective in practice. The choice would depend on the problem at hand, the possible existence of an adversary, the requirement (or lack thereof) for formal guarantees, *etc*. We cannot claim that our approach is better in every way; we show through this work that LRPG is a useful approach for learning policies in control problems where the noise sources are unknown and *we need to retain certain formal guarantees* of the algorithms used. However, training against adversarial noise would possibly yield higher robustness if tuned properly on many problems. An interesting direction would be to prove preservation of guarantees for adversarial noise losses.

Robustness, Complexity and Invariances. Sections 2 and 3 discuss at large the structure, shape and dependence of the maximally robust policy sets. These insights help derive optimisation objectives to use in LRPG, but there is more to be said about how policy robustness is affected by the underlying MDP properties. We hint at this in the proof of Corollary 3.5. More regular (*less complex* in entropy terms, or more *symmetric*) reward functions (*e.g.*, reward functions with smaller variance across the actions $R(x, \cdot, y)$) seem to induce larger robust policy sets. In other words, for a fixed policy, a *more complex* reward function yields larger robustness regrets as soon as any noise is introduced in the system. This raises questions on how to use these principles to derive more robust policies in a comprehensive way, but we leave these questions for future work.

[7] it even outperforms in some cases, although this is probably highly problem dependent, so we do not claim an improvement for every DOMDP

References

Mohammed Amin Abdullah, Hang Ren, Haitham Bou Ammar, Vladimir Milenkovic, Rui Luo, Mingtian Zhang, and Jun Wang. Wasserstein robust reinforcement learning. *arXiv preprint arXiv:1907.13196*, 2019.

Alekh Agarwal, Sham M Kakade, Jason D Lee, and Gaurav Mahajan. On the theory of policy gradient methods: Optimality, approximation, and distribution shift. *J. Mach. Learn. Res.*, 22(98): 1–76, 2021.

Vahid Behzadan and Arslan Munir. Vulnerability of deep reinforcement learning to policy induction attacks. In *International Conference on Machine Learning and Data Mining in Pattern Recognition*, pages 262–275. Springer, 2017.

A. Ben-Israel and B. Mond. What is invexity? *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics*, 28(1):1–9, 1986a.

Adi Ben-Israel and Bertram Mond. What is invexity? *The ANZIAM Journal*, 28(1):1–9, 1986b.

Dimitri Bertsekas. *Nonlinear Programming*. Athena Scientific, 1999.

Dimitri P Bertsekas. Nonlinear programming. *Journal of the Operational Research Society*, 48(3): 334–334, 1997.

Jalaj Bhandari and Daniel Russo. Global optimality guarantees for policy gradient methods. *arXiv preprint arXiv:1906.01786*, 2019.

Shalabh Bhatnagar, Richard S. Sutton, Mohammad Ghavamzadeh, and Mark Lee. Natural actor–critic algorithms. *Automatica*, 45(11):2471–2482, November 2009. doi: 10.1016/j.automatica.2009.07.008.

Vivek S. Borkar. *Stochastic Approximation*. Hindustan Book Agency, 2008.

V.S. Borkar and K. Soumyanatha. An analog scheme for fixed point computation. i. theory. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 44(4):351–355, 1997. doi: 10.1109/81.563625.

Greg Brockman, Vicki Cheung, Ludwig Pettersson, Jonas Schneider, John Schulman, Jie Tang, and Wojciech Zaremba. Openai gym. *arXiv preprint arXiv:1606.01540*, 2016.

Maxime Chevalier-Boisvert, Lucas Willems, and Suman Pal. Minimalistic gridworld environment for openai gym. `https://github.com/maximecb/gym-minigrid`, 2018.

Esther Derman, Daniel Mankowitz, Timothy Mann, and Shie Mannor. A bayesian approach to robust reinforcement learning. In *Uncertainty in Artificial Intelligence*, pages 648–658. PMLR, 2020.

Michael Everett, Björn Lütjens, and Jonathan P How. Certifiable robustness to adversarial state uncertainty in deep reinforcement learning. *IEEE Transactions on Neural Networks and Learning Systems*, 2021.

Marc Fischer, Matthew Mirman, Steven Stalder, and Martin Vechev. Online robustness training for deep reinforcement learning. *arXiv preprint arXiv:1911.00887*, 2019.

Justin Fu, Katie Luo, and Sergey Levine. Learning robust rewards with adverserial inverse reinforcement learning. In *International Conference on Learning Representations*, 2018.

Adam Gleave, Michael Dennis, Cody Wild, Neel Kant, Sergey Levine, and Stuart Russell. Adversarial policies: Attacking deep reinforcement learning. In *International Conference on Learning Representations*, 2020.

Tuomas Haarnoja, Aurick Zhou, Pieter Abbeel, and Sergey Levine. Soft actor-critic: Off-policy maximum entropy deep reinforcement learning with a stochastic actor. In *International conference on machine learning*, pages 1861–1870. PMLR, 2018.

Morgan A Hanson. On sufficiency of the kuhn-tucker conditions. *Journal of Mathematical Analysis and Applications*, 80(2):545–550, 1981.

Matthias Heger. Consideration of risk in reinforcement learning. In *Machine Learning Proceedings 1994*, pages 105–111. Elsevier, 1994.

Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.

Sandy Huang, Nicolas Papernot, Ian Goodfellow, Yan Duan, and Pieter Abbeel. Adversarial attacks on neural network policies. *arXiv preprint arXiv:1702.02284*, 2017.

H Isermann. Linear lexicographic optimization. *Operations-Research-Spektrum*, 4(4):223–228, 1982.

Richard Klima, Daan Bloembergen, Michael Kaisers, and Karl Tuyls. Robust temporal difference learning for critical domains. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems*, AAMAS '19, page 350–358, Richland, SC, 2019. International Foundation for Autonomous Agents and Multiagent Systems. ISBN 9781450363099.

Vijay R. Konda and John N. Tsitsiklis. Actor-critic algorithms. In S. A. Solla, T. K. Leen, and K. Müller, editors, *Advances in Neural Information Processing Systems 12*, pages 1008–1014. MIT Press, 2000.

Jernej Kos and Dawn Song. Delving into adversarial attacks on deep policies. *arXiv preprint arXiv:1705.06452*, 2017.

Yen-Chen Lin, Zhang-Wei Hong, Yuan-Hong Liao, Meng-Li Shih, Ming-Yu Liu, and Min Sun. Tactics of adversarial attack on deep reinforcement learning agents. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence*, pages 3756–3762, 2017.

Boyi Liu, Qi Cai, Zhuoran Yang, and Zhaoran Wang. Neural trust region/proximal policy optimization attains globally optimal policy. In *Advances in Neural Information Processing Systems*, volume 32. Curran Associates, Inc., 2019.

Ajay Mandlekar, Yuke Zhu, Animesh Garg, Li Fei-Fei, and Silvio Savarese. Adversarially robust policy learning: Active construction of physically-plausible perturbations. In *2017 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, pages 3932–3939. IEEE, 2017.

Volodymyr Mnih, Adria Puigdomenech Badia, Mehdi Mirza, Alex Graves, Timothy Lillicrap, Tim Harley, David Silver, and Koray Kavukcuoglu. Asynchronous methods for deep reinforcement learning. In *International conference on machine learning*, pages 1928–1937. PMLR, 2016.

Janosch Moos, Kay Hansel, Hany Abdulsamad, Svenja Stark, Debora Clever, and Jan Peters. Robust reinforcement learning: A review of foundations and recent advances. *Machine Learning and Knowledge Extraction*, 4(1):276–315, 2022.

Jun Morimoto and Kenji Doya. Robust reinforcement learning. *Neural computation*, 17(2):335–359, 2005.

Andrew Y Ng, Daishi Harada, and Stuart Russell. Policy invariance under reward transformations: Theory and application to reward shaping. In *Proc. of the Sixteenth International Conference on Machine Learning, 1999*, 1999.

Arnab Nilim and Laurent El Ghaoui. Robust control of markov decision processes with uncertain transition matrices. *Operations Research*, 53(5):780–798, 2005.

Xinlei Pan, Daniel Seita, Yang Gao, and John Canny. Risk averse robust adversarial reinforcement learning. In *2019 International Conference on Robotics and Automation (ICRA)*, pages 8522–8528. IEEE, 2019.

Santiago Paternain, Luiz F. O. Chamon, Miguel Calvo-Fullana, and Alejandro Ribeiro. Constrained reinforcement learning has zero duality gap. In *Proceedings of the 33rd International Conference on Neural Information Processing Systems*, pages 7553–7563, 2019.

Anay Pattanaik, Zhenyi Tang, Shuijing Liu, Gautham Bommannan, and Girish Chowdhary. Robust deep reinforcement learning with adversarial attacks. *arXiv preprint arXiv:1712.03632*, 2017.

Anay Pattanaik, Zhenyi Tang, Shuijing Liu, Gautham Bommannan, and Girish Chowdhary. Robust deep reinforcement learning with adversarial attacks. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems*, AAMAS '18, page 2040–2042, Richland, SC, 2018. International Foundation for Autonomous Agents and Multiagent Systems.

Lerrel Pinto, James Davidson, Rahul Sukthankar, and Abhinav Gupta. Robust adversarial reinforcement learning. In *International Conference on Machine Learning*, pages 2817–2826. PMLR, 2017.

Matteo Pirotta, Marcello Restelli, Alessio Pecorino, and Daniele Calandriello. Safe policy iteration. In *International Conference on Machine Learning*, pages 307–315. PMLR, 2013.

Antonin Raffin, Ashley Hill, Adam Gleave, Anssi Kanervisto, Maximilian Ernestus, and Noah Dormann. Stable-baselines3: Reliable reinforcement learning implementations. *Journal of Machine Learning Research*, 22(268):1–8, 2021. URL http://jmlr.org/papers/v22/20-1364.html.

Mark J Rentmeesters, Wei K Tsai, and Kwei-Jay Lin. A theory of lexicographic multi-criteria optimization. In *Proceedings of ICECCS'96: 2nd IEEE International Conference on Engineering of Complex Computer Systems (held jointly with 6th CSESAW and 4th IEEE RTAW)*, pages 76–79. IEEE, 1996.

Jay K Satia and Roy E Lave Jr. Markovian decision processes with uncertain transition probabilities. *Operations Research*, 21(3):728–740, 1973.

John Schulman, Filip Wolski, Prafulla Dhariwal, Alec Radford, and Oleg Klimov. Proximal policy optimization algorithms. *arXiv preprint arXiv:1707.06347*, 2017.

Joar Skalse, Matthew Farrugia-Roberts, Stuart Russell, Alessandro Abate, and Adam Gleave. Invariance in policy optimisation and partial identifiability in reward learning. *arXiv preprint arXiv:2203.07475*, 2022a.

Joar Skalse, Lewis Hammond, Charlie Griffin, and Alessandro Abate. Lexicographic multi-objective reinforcement learning. In *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence*, pages 3430–3436, jul 2022b. doi: 10.24963/ijcai.2022/476.

Morton Slater. Lagrange multipliers revisited. Cowles Commission Discussion Paper No. 403, 1950.

Matthijs TJ Spaan. Partially observable markov decision processes. In *Reinforcement Learning*, pages 387–414. Springer, 2012.

Matthijs TJ Spaan and N Vlassis. A point-based pomdp algorithm for robot planning. In *IEEE International Conference on Robotics and Automation, 2004. Proceedings. ICRA'04. 2004*, volume 3, pages 2399–2404. IEEE, 2004.

Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

Richard S. Sutton, David McAllester, Satinder Singh, and Yishay Mansour. Policy gradient methods for reinforcement learning with function approximation. In *Proceedings of the 12th International Conference on Neural Information Processing Systems*, NIPS'99, pages 1057–1063, Cambridge, MA, USA, 1999. MIT Press.

Kai Liang Tan, Yasaman Esfandiari, Xian Yeow Lee, Soumik Sarkar, et al. Robustifying reinforcement learning agents via action space adversarial training. In *2020 American control conference (ACC)*, pages 3959–3964. IEEE, 2020.

Chen Tessler, Yonathan Efroni, and Shie Mannor. Action robust reinforcement learning and applications in continuous control. In *International Conference on Machine Learning*, pages 6215–6224. PMLR, 2019.

Elise van der Pol, Thomas Kipf, Frans A. Oliehoek, and Max Welling. Plannable approximations to mdp homomorphisms: Equivariance under actions. In *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems*, AAMAS '20, page 1431–1439, Richland, SC, 2020. International Foundation for Autonomous Agents and Multiagent Systems. ISBN 9781450375184.

Bart PG Van Parys, Daniel Kuhn, Paul J Goulart, and Manfred Morari. Distributionally robust control of constrained stochastic systems. *IEEE Transactions on Automatic Control*, 61(2):430–442, 2015.

Wolfram Wiesemann, Daniel Kuhn, and Melvyn Sim. Distributionally robust convex optimization. *Operations Research*, 62(6):1358–1376, 2014.

Huan Xu and Shie Mannor. The robustness-performance tradeoff in markov decision processes. *Advances in Neural Information Processing Systems*, 19, 2006.

Huan Zhang, Hongge Chen, Chaowei Xiao, Bo Li, Mingyan Liu, Duane Boning, and Cho-Jui Hsieh. Robust deep reinforcement learning against adversarial perturbations on state observations. *Advances in Neural Information Processing Systems*, 33:21024–21037, 2020.

Huan Zhang, Hongge Chen, Duane Boning, and Cho-Jui Hsieh. Robust reinforcement learning on state observations with learned optimal adversary. In *International Conference on Learning Representation (ICLR)*, 2021.

Shangtong Zhang. Modularized implementation of deep rl algorithms in pytorch. `https://github.com/ShangtongZhang/DeepRL`, 2018.

A Examples and Further Considerations

We provide here two examples to show how we can obtain limit scenarios $\Pi_0 = \Pi$ (any policy is maximally robust) or $\Pi_0 = \Pi_T$ (Example 1), and how for some MDPs the third inclusion in Theorem 3.4 is strict (Example 2).

Example 1 Consider the simple MDP in Figure 2. First, consider the reward function $R_1(x_1, \cdot, \cdot) = 10$, $R_1(x_2, \cdot, \cdot) = 0$. This produces a "dummy" MDP where all policies have the same reward sum. Then, $\forall T, \pi$, $V^{\langle \pi, T\rangle} = V^\pi$, and therefore we have $\Pi_D = \Pi_0 = \Pi$.

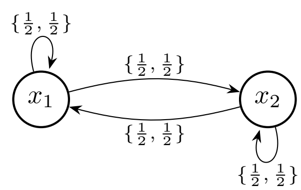


Figure 2: Example MDP. Values in brackets represent $\{P(\cdot, u_1, \cdot), P(\cdot, u_2, \cdot)\}$.

Now, consider the reward function $R_2(x_1, u_1, \cdot) = 10$, $R_2(\cdot, \cdot, \cdot) = 0$ elsewhere. Take a non-constant policy π, *i.e.*, $\pi(x_1) \neq \pi(x_2)$. In the example DOMDP (assuming the initial state is drawn uniformly from $X_0 = \{x_1, x_2\}$) one can show that at any time in the trajectory, there is a stationary probability $\Pr\{x_t = x_1\} = \frac{1}{2}$. Let us abuse notation and write $\pi(x_i) = \begin{pmatrix} \pi(x_i, u_1) & \pi(x_i, u_2) \end{pmatrix}^\top$ and $R(x_i) = \begin{pmatrix} R(x_i, u_1, \cdot) & R(x_i, u_2, \cdot) \end{pmatrix}^\top$. For the given reward structure we have $R(x_2) = \begin{pmatrix} 0 & 0 \end{pmatrix}^\top$, and therefore:

$$J(\pi) = E_{x_0 \sim \mu_0}\left[\sum_{t=0}^{\infty} \gamma^t R_t\right] = \frac{1}{2}\langle R(x_1), \pi(x_1)\rangle \frac{\gamma}{1-\gamma}. \tag{7}$$

Since the transitions of the MDP are independent of the actions, following the same principle as in (7): $J\langle \pi, T\rangle = \frac{1}{2}\langle R(x_1), \langle \cdot, T\rangle(\pi)(x_1)\rangle \frac{\gamma}{1-\gamma}$. For any noise map $\langle \cdot, T\rangle \neq \mathrm{Id}$, for the two-state policy it holds that $\pi \notin \Pi_T \implies \langle \pi, T\rangle \neq \pi$. Therefore $\langle \pi, T\rangle(x_1) \neq \pi(x_1)$ and:

$$J(\pi) - J(\langle \pi, T\rangle) = \frac{5\gamma}{1-\gamma} \cdot \big(\pi(x_1, 1) - \langle \pi, T\rangle(x_1, 1)\big) \neq 0,$$

which implies that $\pi \notin \Pi_0$.

Example 2 Consider the same MDP in Figure 2 with reward function $R(x_1, u_1, \cdot) = R(x_2, u_1, \cdot) = 10$, and a reward of zero for all other transitions. Take a policy $\pi(x_1) = (1\ 0)$, $\pi(x_2) = (0\ 1)$. The policy yields a reward of 10 in state x_1 and a reward of 0 in state x_2. Again we assume the initial state is drawn uniformly from $X_0 = \{x_1, x_2\}$. Then, observe:

$$J(\pi) = E_{x_0 \sim \mu_0}\left[\sum_{t=0}^{\infty} \gamma^t R_t\right] = \frac{1}{2}\langle R(x_1), \pi(x_1)\rangle \frac{\gamma}{1-\gamma} = \frac{1}{2}\frac{10\gamma}{1-\gamma} = \frac{5\gamma}{1-\gamma}.$$

Define now noise map $T(\cdot \mid x_1) = (\frac{1}{2}\ \frac{1}{2})$ and $T(\cdot \mid x_2) = (\frac{1}{2}\ \frac{1}{2})$. Observe this noise map yields a policy with non-zero disadvantage, $D^\pi(x_1, T) = \frac{5\gamma}{1-\gamma} - \big(\frac{5\gamma}{1-\gamma} - 2.5\big) = 2.5$ and similarly $D^\pi(x_2, T) = -2.5$, therefore $\pi \notin \Pi_D$. However, the policy *is maximally robust*:

$$J(\langle \pi, T\rangle) = \frac{1}{2}\langle R(x_1), \langle \pi, T\rangle(x_1)\rangle \frac{\gamma}{1-\gamma} + \frac{1}{2}\langle R(x_2), \langle \pi, T\rangle(x_2)\rangle \frac{\gamma}{1-\gamma} = \frac{1}{2}\frac{\gamma}{1-\gamma}\big(5+5\big) = \frac{5\gamma}{1-\gamma}. \tag{8}$$

Therefore, $\pi \in \Pi_0$.

B Theoretical Results

B.1 Auxiliary Results

Theorem B.1 (Stochastic Approximation with Non-Expansive Operator)**.** *Let $\{\xi_t\}$ be a random sequence with $\xi_t \in \mathbb{R}^n$ defined by the iteration:*

$$\xi_{t+1} = \xi_t + \alpha_t(F(\xi_t) - \xi_t + M_{t+1}),$$

where:

1. *The step sizes α_t satisfy Assumption 2.*
2. *$F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a $\|\cdot\|_\infty$ non-expansive map. That is, for any $\xi_1, \xi_2 \in \mathbb{R}^n$, $\|F(\xi_1) - F(\xi_2)\|_\infty \leq \|\xi_1 - \xi_2\|_\infty$.*
3. *$\{M_t\}$ is a martingale difference sequence with respect to the increasing family of $\sigma-$fields $\mathcal{F}_t := \sigma(\xi_0, M_0, \xi_1, M_1, ..., \xi_t, M_t)$.*

Then, the sequence $\xi_t \to \xi^*$ almost surely *where ξ^* is a fixed point such that $F(\xi^*) = \xi^*$.*

Proof. See Borkar and Soumyanatha [1997]. □

Theorem B.2 (PB-LRL Convergence)**.** *Let $\mathcal{M}$ be a multi-objective MDP with objectives K_i, $i \in \{1, ..., m\}$ of the same form. Assume a policy π is twice differentiable in parameters θ, and if using a critic V_i assume it is continuously differentiable on w_i. Suppose that if PB-LRL is run for T steps, there exists some limit point $w_i^*(\theta)$ when θ is held fixed under conditions $\mathcal{C}$ on $\mathcal{M}$, π and V_i. If $\lim_{T\to\infty} \mathbb{E}_t[\theta] \in \Theta_1^\epsilon$ for $m = 1$, then for any $m \in \mathbb{N}$ we have $\lim_{T\to\infty} \mathbb{E}_t[\theta] \in \Theta_m^\epsilon$ where ϵ depends on the representational power of the parameterisations of π, V_i.*

Proof Sketch. We refer the interested reader to Skalse et al. [2022b] for a full proof, and here attempt to provide the intuition behind the result in the form of a proof sketch.

Let us begin by briefly recalling the general problem statement: we wish to take a multi-objective MDP $\mathcal{M}$ with m objectives, and obtain a lexicographically optimal policy (one that optimises the first objective, and then subject to this optimises the second objective, and so on). More precisely, for a policy π parameterised by θ, we say that π is (globally) *lexicographically ϵ-optimal* if $\theta \in \Theta_m^\epsilon$, where $\Theta_0^\epsilon = \Theta$ is the set of all policies in $\mathcal{M}$, $\Theta_{i+1}^\epsilon := \{\theta \in \Theta_i^\epsilon \mid \max_{\theta' \in \Theta_i^\epsilon} K_i(\theta') - K_i(\theta) \leq \epsilon_i\}$, and $\mathbb{R}^{m-1} \ni \epsilon \succcurlyeq 0$.[8]

The basic idea behind policy-based lexicographic reinforcement learning (PB-LRL) is to use a multi-timescale approach to first optimise θ using K_1, then at a slower timescale optimise θ using K_2 while adding the condition that the loss with respect to K_1 remains bounded by its current value, and so on. This sequence of constrained optimisations problems can be solved using a Lagrangian relaxation [Bertsekas, 1999], either in series or – via a judicious choice of learning rates – simultaneously, by exploiting a separation in timescales [Borkar, 2008]. In the simultaneous case, the parameters of the critic w_i (if using an actor-critic algorithm, if not this part of the argument may be safely ignored) for each objective are updated on the fastest timescale, then the parameters θ, and finally (i.e., most slowly) the Lagrange multipliers for each of the remaining constraints.

The proof proceeds via induction on the number of objectives, using a standard stochastic approximation argument [Borkar, 2008]. In particular, due to the learning rates chosen, we may consider those more slowly updated parameters fixed for the purposes of analysing the convergence of the more quickly updated parameters. In the base case where $m = 1$, we have (by assumption) that $\lim_{T\to\infty} \mathbb{E}_t[\theta] \in \Theta_1^\epsilon$. This is simply the standard (non-lexicographic) RL setting. Before continuing to the inductive step, Skalse et al. [2022b] observe that because gradient descent on K_1 converges to globally optimal stationary point when $m = 1$ then K_1 must be globally *invex* (where the opposite implication is also true) [Ben-Israel and Mond, 1986a].[9]

[8] The proof in Skalse et al. [2022b] also considers *local* lexicographic optima, though for the sake of simplicity, we do not do so here.

[9] A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is (globally) invex if and only if there exists a function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x_1) - f(x_2) \geq g(x_1, x_2)^\top \nabla f(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$ [Hanson, 1981].

The reason this observation is useful is that because each of the objectives K_i shares the same functional form, they are all invex, and furthermore, invexity is conserved under linear combinations and the addition of scalars, meaning that the Lagrangian formed in the relaxation of each constrained optimisation problem is also invex. As a result, if we assume that $\lim_{T\to\infty}\mathbb{E}_t[\theta] \in \Theta_i^\epsilon$ as our inductive hypothesis, then the stationary point of the Lagrangian for optimising objective K_{i+1} is a global optimum, given the constraints that it does not worsen performance on $K_1,\ldots,K_i$. Via Slater's condition [Slater, 1950] and standard saddle-point arguments [Bertsekas, 1999, Paternain et al., 2019], we therefore have that $\lim_{T\to\infty}\mathbb{E}_t[\theta] \in \Theta_{i+1}^\epsilon$, completing the inductive step, and thus the overall inductive argument.

This concludes the proof that $\lim_{T\to\infty}\mathbb{E}_t[\theta] \in \Theta_m^\epsilon$. We refer the reader to Skalse et al. [2022b] for a discussion of the error ϵ, but intuitively it corresponds to a combination of the representational power of θ, the critic parameters w_i (if used), and the duality gap due to the Lagrangian relaxation [Paternain et al., 2019]. In cases where the representational power of the various parameters is sufficiently high, then it can be shown that $\epsilon = 0$. ☐

Lemma B.3. *Let π_θ be a fully-parameterised policy in a DOMDP, and α_t a learning rate satisfying Assumption 2. Consider the following* approximated *gradient for objective $K_{\tilde{T}}(\pi)$ and sampled point $x \in X$:*

$$\nabla_\theta \hat{K}_{\tilde{T}}(\theta)(x) = (\pi_\theta(x) - \pi_\theta(y))\nabla_\theta \pi_\theta(x), \quad y \sim \tilde{T}(\cdot \mid x). \tag{9}$$

Then, the following iteration with $x \in X$ and some initial θ_0,

$$\theta_{t+1} = \theta_t - \alpha_t \nabla_\theta \hat{K}_{\tilde{T}}(\theta_t) \tag{10}$$

yields $\theta \to \tilde{\theta}$ almost surely *where $\tilde{\theta}$ satisfies $K_{\tilde{T}}(\tilde{\theta}) = 0$.*

Lemma B.3. We make use of standard results on stochastic approximation with non-expansive operators (specifically, Theorem B.1 in the appendix) Borkar and Soumyanatha [1997]. First, observe that for a fully parameterised policy, one can assume to have a tabular representation such that $\pi_\theta(x,u) = \theta_{xu}$, and $\nabla_\theta \pi_\theta(x) \equiv \mathrm{Id}$. We can then write the stochastic gradient descent problem in terms of the policy. Let $y \sim \tilde{T}(\cdot \mid x)$. Then:

$$\begin{aligned}\pi_{t+1}(x) &= \pi_t(x) - \alpha_t\big(\pi_t(x) - \pi_t(y)\big) = \\ &= \pi_t(x) - \alpha_t\Big(\pi_t(x) - \langle \pi_t, \tilde{T}\rangle(x) - \big(\pi_t(y) - \langle \pi_t, \tilde{T}\rangle(x)\big)\Big).\end{aligned}$$

We now need to verify that the necessary conditions for applying Theorem B.1 hold. First, α_t satisfies Assumption 2. Second, making use of the property $\|\tilde{T}\|_\infty = 1$ for any row-stochastic matrix $\tilde{T}$, for any two policies $\pi_1, \pi_2 \in \Pi$:

$$\|\langle \pi_1, \tilde{T}\rangle - \langle \pi_2, \tilde{T}\rangle\|_\infty = \|\tilde{T}\pi_1 - \tilde{T}\pi_2\|_\infty = \|\tilde{T}(\pi_1 - \pi_2)\|_\infty \leq \|\tilde{T}\|_\infty \|\pi_1 - \pi_2\|_\infty = \|\pi_1 - \pi_2\|_\infty.$$

Therefore, the operator $\langle \cdot, \tilde{T}\rangle$ is non-expansive with respect to the sup-norm. For the final condition, we have

$$\mathbb{E}_{y\sim\tilde{T}(\cdot|x)}\left[\pi_t(y) - \langle \pi_t, \tilde{T}\rangle(x) \mid \pi_t, \tilde{T}\right] = \sum_{y\in X} \tilde{T}(y \mid x)\pi_t(y) - \langle \pi_t, \tilde{T}\rangle(x) = 0.$$

Therefore, the difference $\pi_t(y) - \langle \pi_t, \tilde{T}\rangle(x)$ is a martingale difference for all x. One can then apply Theorem B.1 with $\xi_t(x) \equiv \pi_t(x)$, $F(\cdot) \equiv \langle \cdot, \tilde{T}\rangle$ and $M_{t+1} \equiv \pi_t(y) - \langle \pi_t, \tilde{T}\rangle(x)$ to conclude that $\pi_t(x) \to \tilde{\pi}(x)$ almost surely. Finally from assumption 2.2, for any policy all states $x \in X$ are visited infinitely often, therefore $\pi_t(x) \to \tilde{\pi}(x) \forall x \in X \implies \pi_t \to \tilde{\pi}$ and $\tilde{\pi}$ satisfies $\langle \tilde{\pi}, \tilde{T}\rangle = \tilde{\pi}$, and $K_{\tilde{T}}(\tilde{\pi}) = 0$. ☐

B.2 Proofs

We now present the proofs for the statements through the work.

Proposition 3.2. If a policy $\pi \in \Pi$ is a fixed point of the operator $\langle \cdot, T\rangle$, then it holds that $\langle \pi, T\rangle = \pi$. Therefore, one can compute the robustness of the policy π to obtain $\rho(\pi, T) = J(\pi) - J(\langle \pi, T\rangle) = J(\pi) - J(\pi) = 0 \implies \pi \in \Pi_0$. Therefore, $\Pi_T \subseteq \Pi_0$.

For a discrete state and action spaces, the space of stochastic kernels $\mathcal{K} : X \mapsto \Delta(X)$ is equivalent to the space of row-stochastic $|X| \times |X|$ matrices, therefore one can write $T(y \mid x) \equiv T_{xy}$ as the $xy-$th entry of the matrix T. Then, the representation of a constant policy as an $X \times U$ matrix can be written as $\overline{\pi} = \mathbf{1}_{|X|} v^\top$, where $\mathbf{1}_{|X|}$ where $v \in \Delta(U)$ is any probability distribution over the action space. Observe that, applying the operator $\langle \pi, T \rangle$ to a constant policy yields:

$$\langle \overline{\pi}, T \rangle = T \mathbf{1}_{|X|} v^\top. \tag{11}$$

By the Perron-Frobenius Theorem [Horn and Johnson, 2012], since T is row-stochastic it has at least one eigenvalue $\mathrm{eig}(T) = 1$, and this admits a (strictly positive) eigenvector $T\mathbf{1}_{|X|} = \mathbf{1}_{|X|}$. Therefore, substituting this in (11):

$$\langle \overline{\pi}, T \rangle = T \mathbf{1}_{|X|} v^\top = \mathbf{1}_{|X|} v^\top = \overline{\pi} \implies \overline{\Pi} \subseteq \Pi_T.$$

$\square$

Proposition 3.3. Recall the definition in (2) and that the noise disadvantage function of a policy π is given by (4). We want to show that $D^\pi(x, T) = 0 \implies \rho(\pi, T) = 0$. Taking $D^\pi(x, T) = 0$ one has a policy that produces an disadvantage of zero when noise kernel T is applied. Then,

$$D^\pi(x, T) = 0 \implies \mathbb{E}_{u \sim \langle \pi, T \rangle (x)}[Q^\pi(x, u)] = V^\pi(x) \ \forall \, x \in X. \tag{12}$$

Now define the value of the disturbed policy

$$V^{\langle \pi, T \rangle}(x_0) := \mathbb{E}_{\substack{u_k \sim \langle \pi, T \rangle (x_k), \\ x_{k+1} \sim P(\cdot | x_k, u_k)}} \left[\sum_{k=0}^\infty \gamma^k r(x_k, u_k) \right],$$

and take:

$$V^{\langle \pi, T \rangle}(x) = \mathbb{E}_{\substack{u \sim \langle \pi, T \rangle (x), \\ y \sim P(\cdot | x, u)}} \left[r(x, u, y) + \gamma V^{\langle \pi, T \rangle}(y) \right].$$

We will now show that $V^\pi(x) = V^{\langle \pi, T \rangle}(x)$, for all $x \in X$. Observe, from (12) using $V^\pi(x) = \mathbb{E}_{u \sim \langle \pi, T \rangle (x)}[Q^\pi(x, u)]$, we have $\forall x \in X$:

$$\begin{aligned} V^\pi(x) - V^{\langle \pi, T \rangle}(x) = & \mathbb{E}_{u \sim \langle \pi, T \rangle (x)}[Q^\pi(x, u)] - \mathbb{E}_{\substack{u \sim \langle \pi, T \rangle (x) \\ y \sim P(\cdot | x, u)}} \left[r(x, u, y) + \gamma V^{\langle \pi, T \rangle}(y) \right] \\ = & \mathbb{E}_{\substack{u \sim \langle \pi, T \rangle (x) \\ y \sim P(\cdot | x, u)}} \left[r(x, u, y) + \gamma V^\pi(y) - r(x, u, y) - \gamma V^{\langle \pi, T \rangle}(y) \right] \\ = & \gamma \mathbb{E}_{y \sim P(\cdot | x, u)} \left[V^\pi(y) - V^{\langle \pi, T \rangle}(y) \right]. \end{aligned} \tag{13}$$

Now, taking the sup norm at both sides of (13) we get

$$\| V^\pi(x) - V^{\langle \pi, T \rangle}(x) \|_\infty = \gamma \left\| \mathbb{E}_{y \sim P(\cdot | x, u)} \left[V^\pi(y) - V^{\langle \pi, T \rangle}(y) \right] \right\|_\infty. \tag{14}$$

Observe that for the right hand side of (14), we have $\left\| \mathbb{E}_{y \sim P(\cdot | x, u)} \left[V^\pi(y) - V^{\langle \pi, T \rangle}(y) \right] \right\|_\infty \leq \| V^\pi(x) - V^{\langle \pi, T \rangle}(x) \|_\infty$. Therefore, since $\gamma < 1$,

$$\| V^\pi(x) - V^{\langle \pi, T \rangle}(x) \|_\infty \leq \gamma \| V^\pi(x) - V^{\langle \pi, T \rangle}(x) \|_\infty \implies \| V^\pi(x) - V^{\langle \pi, T \rangle}(x) \|_\infty = 0. \tag{15}$$

Finally, $\| V^\pi(x) - V^{\langle \pi, T \rangle}(x) \|_\infty = 0 \implies V^\pi(x) - V^{\langle \pi, T \rangle}(x) = 0 \ \forall x \in X$, and $V^\pi(x) - V^{\langle \pi, T \rangle}(x) = 0 \, \forall \, x \in X \implies J(\pi) = J(\langle \pi, T \rangle) \implies \rho(\pi, T) = 0$. $\square$

Inclusion Theorem 3.4. Combining Proposition 3.2 and Proposition 3.3, we simply need to show that $\Pi_T \subset \Pi_D$. Take π to be a fixed point of $\langle \pi, T \rangle$. Then $\langle \pi, T \rangle = \pi$, and from the definition in (4):

$$\begin{aligned} D^\pi(x, T) = & V^\pi(x) - \mathbb{E}_{u \sim \langle \pi, T \rangle (x, \cdot)}[Q^\pi(x, u)] \\ = & V^\pi(x) - \mathbb{E}_{u \sim \pi(x, \cdot)}[Q^\pi(x, u)] \\ = & V^\pi(x) - V^\pi(x) = 0. \end{aligned}$$

Therefore, $\pi \in \Pi_D$, which completes the sequence of inclusions.

To show convexity of $\overline{\Pi}, \Pi_T$, first for a constant policy $\overline{\pi} \in \overline{\Pi}$, recall that we can write $\overline{\pi} = \mathbf{1}v^\top$, where $v \in \Delta(U)$ is any probability distribution over the action space. Now take $\overline{\pi}_1, \overline{\pi}_2 \in \overline{\Pi}$. For any $\alpha \in [0,1]$, $\alpha\overline{\pi}_1 + (1-\alpha)\overline{\pi}_2 = \alpha\mathbf{1}v_1^\top + (1-\alpha)\mathbf{1}v_2^\top = \mathbf{1}(\alpha v_1 + (1-\alpha)v_2)^\top \in \overline{\Pi}$.

At last, for the set Π_T, assume there exist two different policies π_1, π_2 both fixed points of $\langle\cdot, T\rangle$. Then, for any $\alpha \in [0,1]$, $\langle(\alpha\pi_1 + (1-\alpha)\pi_2), T\rangle = \alpha T\pi_1 + (1-\alpha)T\pi_2 = \alpha\pi_1 + (1-\alpha)\pi_2$. Therefore, any affine combination of fixed points is also a fixed point. ☐

Corollary 3.5. For statement (i), let $\overline{R}(\cdot,\cdot,\cdot) = c$ for some constant $c \in \mathbb{R}$. Then, $J(\pi) = \mathbb{E}_{x_0\sim\mu_0}[\sum_t \gamma^t \overline{r}_t \mid \pi] = \frac{c\gamma}{1-\gamma}$, which does not depend on the policy π. For any noise kernel T and policy π, $J(\pi) - J\langle\pi, T\rangle = 0 \implies \pi \in \Pi_0$.

For statement (ii) assume $\exists \pi \in \Pi_0 : \pi \notin \Pi_T$. Then, $\exists x^* \in X$ and $u^* \in U$ such that $\pi(x^*, u^*) \neq \langle\pi, T\rangle(x^*, u^*)$. Let:

$$\underline{R}(x, u, x') := \begin{cases} c & \text{if } x = x^* \text{ and } u = u^* \\ 0 & \text{otherwise} \end{cases}.$$

Then, $\mathbb{E}[R(x, \pi(x), x'] < \mathbb{E}[R(x, \langle\pi, T\rangle(x), x']$ and since the MDP is ergodic x is visited infinitely often and $J(\pi) - J(\langle\pi, T\rangle) > 0 \implies \pi \notin \Pi_0$, which contradicts the assumption. Therefore, $\Pi_0 \setminus \Pi_T = \emptyset \implies \Pi_0 = \Pi_T$. ☐

Theorem 4.2. We apply the results from Skalse et al. [2022b] in Theorem B.2. Essentially, Skalse et al. [2022b] prove that for a policy gradient algorithm to lexicographically optimise a policy for multiple objectives, it is a sufficient condition that the stochastic gradient descent algorithm finds optimal parameters for each of the objectives independently. From Lemma B.3 we know that a policy gradient algorithm using the gradient estimate in (9) converges to a maximally robust policy, *i.e.* a set of parameters $\theta' = \operatorname{argmax}_\theta K_{\tilde{T}}$. Additionally, by assumption, the chosen algorithm for K_1 converges to an optimal point θ^*. While the two objective functions are not of the same form – as in Skalse et al. [2022b] – the fact they are both invex [Ben-Israel and Mond, 1986b] either locally or globally depending on the form of K_1, implies that $\hat{K}$ is also invex and hence that the stationary point θ^ϵ computed by LRPG satisfies equation 6. ☐

Corollary 5.1. The proof follows by the inclusion results in Theorem 3.4. If $\Pi_{\tilde{T}} = \overline{\Pi}$, then $\Pi_{\tilde{T}} \subseteq \Pi_T$ for any other T. Then, the distance from π to the set Π_T is at most the distance to $\Pi_{\tilde{T}}$. ☐

B.3 On Adversarial Disturbances and other Noise Kernels

A problem that remains open after this work is what constitutes an appropriate choice of $\tilde{T}$, and what can we expect by restricting a particular class of $\tilde{T}$. We first discuss adversarial examples, and then general considerations on $\tilde{T}$ versus T.

Adversarial Noise As mentioned in the introduction, much of the previous work focuses on adversarial disturbances. We did not directly address this in the results of this work since our motivation lies in the scenarios where the disturbance is not adversarial and is unknown. However, following the results of Section 3, we are able to reason about adversarial disturbances. Consider an adversarial map T_{adv} to be

$$\langle\pi, T_{adv}\rangle(x) = \pi(y), \quad y \in \operatorname{argmax}_{y\in X_{ad}(x)} d\big(\pi(x), \pi(y)\big),$$

with $X_{ad}(x) \subseteq X$ being a set of admissible disturbance states for x, and $d(\cdot,\cdot)$ is a distance measure between distributions (*e.g.* 2-norm).

Proposition B.4. *Constant policies are a fixed point of T_{adv}, and are the only fixed points if for all pairs x_0, x_k there exists a sequence $\{x_0, ..., x_k\} \subseteq X$ such that $x_i \in X_{ad}(x_i)$.*

Proof. First, it is straight-forward that if $\overline{\pi} \in \overline{\Pi} \implies \langle\overline{\pi}, T_{adv}\rangle(x) = \overline{\pi}(x)$. To show they are the only fixed points, assume that there is a non-constant policy π' that is a fixed point of T_{ad}. Then, there exists x, z such that $\pi'(x) \neq \pi'(z)$. However, by assumption, we can construct a sequence $\{x, ..., z\} \subseteq X$ that connects x and z and every state in the sequence is in the admissible set of the previous one. Assume without loss of generality that this sequence is $\{x, y, z\}$. Then, if π' is a fixed point, $\langle \pi', T_{adv} \rangle(x) = \pi'(x)$, $\langle \pi', T_{adv} \rangle(y) = \pi'(y)$ and $\langle \pi', T_{adv} \rangle(z) = \pi'(z)$. However, $\pi'(x) \neq \pi'(z)$, so either $\pi'(x) \neq \pi'(y) \implies d(\pi'(x), \pi'(y)) \neq 0$ or $\pi'(y) \neq \pi'(z) \implies d(\pi'(y), \pi'(z)) \neq 0$, therefore π' cannot be a fixed point of T_{adv}. $\square$

The main difference between an adversarial operator and the random noise considered throughout this work is that T_{adv} is *not a linear operator*, and additionally, it is time varying (since the policy is being modified at every time step of the PG algorithm). Therefore, including it as a LRPG objective would invalidate the assumptions required for LRPG to retain formal guarantees of the original PG algorithm used, and it is not guaranteed that the resulting policy gradient algorithm would converge.

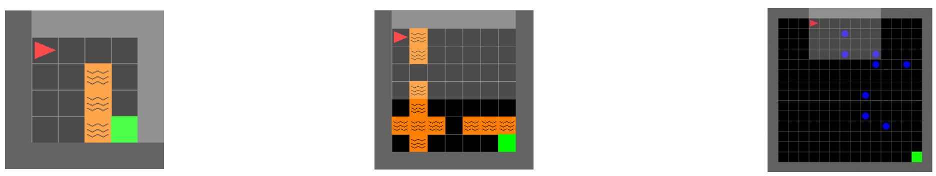
C Experiment Methodology

We use in the experiments well-tested implementations of A2C, PPO and SAC from Stable Baselines 3 [Raffin et al., 2021] to include the computation of the lexicographic parameters in (1).

LRPG Parameters. The LRL parameters are initialised in all cases as $\beta_0^1 = 2$, $\beta_0^2 = 1$, $\lambda = 0$ and $\eta = 0.001$. The LRL tolerance is set to $\epsilon_t = 0.99\hat{k}_1$ to ensure we never deviate too much from the original objective, since the environments have very sparse rewards. We use a first order approximation to compute the LRL weights from the original LMORL implementation.

C.1 Discrete Control

The discrete control environments used can be seen in Figure 3. Since all the environments use a



(a) *MiniGrid-LavaGap* (b) *MiniGrid-LavaCrossing* (c) *MiniGrid-DynamicObstacles*

Figure 3: Screenshots of the environments used.

pixel representation of the observation, we use a shared representation for the value function and policy, where the first component is a convolutional network, implemented as in Zhang [2018]. The hyper-parameters of the neural representations are presented in Table 3.

Layer	Output	Func.
Conv1	16	ReLu
Conv2	32	ReLu
Conv3	64	ReLu

Table 3: Shared Observation Layers

The actor and critic layers, for both algorithms, are a fully connected layer with 64 features as input and the corresponding output. We used in all cases an Adam optimiser. We optimised the parameters for each (vanilla) algorithm through a quick parameter search, and apply the same parameters for the Lexicographically Robust versions.

	LavaGap	LavaCrossing	DynamicObstacles
Parallel Envs	16	16	16
Steps	$2 \cdot 10^6$	$2 \cdot 10^6$	8×10^6
γ	0.99	0.99	0.98
α	0.00176	0.00176	0.00181
ϵ(Adam)	10^{-8}	10^{-8}	10^{-8}
Grad. Clip	0.9	0.9	0.5
Gae	0.95	0.95	0.95
Rollout	64	64	64
E. Coeff	0.01	0.014	0.011
V. Coeff	0.05	0.05	0.88

Table 4: A2C Parameters

	LavaGap	LavaCrossing	DynamicObstacles
Parallel Envs	8	8	8
Steps	$6 \cdot 10^6$	$2 \cdot 10^6$	8×10^5
γ	0.95	0.99	0.97
α	0.001	0.001	0.001
ϵ(Adam)	10^{-8}	10^{-8}	10^{-8}
Grad. Clip	1	1	0.1
Ratio Clip	0.2	0.2	0.2
Gae	0.95	0.95	0.95
Rollout	256	512	256
Epochs	10	10	10
E. Coeff	0	0.1	0.01

Table 5: PPO Parameters

For the implementation of the LRPG versions of the algorithms, in all cases we allow the algorithm to iterate for $1/3$ of the total steps before starting to compute the robustness objectives. In other words, we use $\hat{K}(\theta) = K_1(\theta)$ until $t = \frac{1}{3}\,\text{max\_steps}$, and from this point we resume the lexicographic robustness computation as described in Algorithm 1. This is due to the structure of the environments simulated. The rewards (and in particular the positive rewards) are very sparse in the environments considered. Therefore, when computing the policy gradient steps, the loss for the primary objective is practically zero until the environment is successfully solved at least once. If we implement the combined lexicographic loss from the first time step, many times the algorithm would converge to a (constant) policy without exploring for enough steps, leading to convergence towards a maximally robust policy that does not solve the environment.

Noise Kernels. We consider two types of noise; a normal distributed noise $\tilde{T}^g$ and a uniform distributed noise $\tilde{T}^u$. For the environments LavaGap and DynamicObstacles, the kernel $\tilde{T}^u$ produces a disturbed state $\tilde{x} = x + \xi$ where $\|\xi\|_\infty \leq 2$, and for LavaCrossing $\|\xi\|_\infty \leq 1.5$. The normal distributed noise is in all cases $\mathcal{N}(0, 0.5)$. The maximum norm of the noise is quite large, but this is due to the structure of the observations in these environments. The pixel values are encoded as integers $0 - 9$, where each integer represents a different feature in the environment (empty space, doors, lava, obstacle, goal...). Therefore, any noise $\|\xi\|_\infty \leq 0.5$ would most likely not be enough to *confuse* the agent. On the other hand, too large noise signals are unrealistic and produce pathological environments. All the policies are then tested against two “true" noise kernels, $T_1 = \tilde{T}^u$ and $T_2 = \tilde{T}^g$. The main reason for this is to test both the scenarios where we assume a *wrong* noise kernel, and the case where we are training the agents with the correct kernel.

Comparison with SA-PPO. One of the baselines included is the State-Adversarial PPO algorithm proposed in Zhang et al. [2020]. The implementation includes an extra parameter that multiplies the regularisation objective, k_{ppo}. Since we were not able to find indications on the best parameter for discrete action environments, we implemented $k_{ppo} \in \{0.1, 1, 2\}$ and picked the best result for each entry in Table 1. Larger values seemed to de-stabilise the learning in some cases. The rest of the parameters are kept as in the vanilla PPO implementation.

C.1.1 Extended Results: Adversarial Disturbances

Even though we do not use an adversarial attacker or disturbance in our reasoning through this work, we implemented a policy-based state-adversarial noise disturbance to test the benchmark algorithms against, and evaluate how well each of the methods reacts to such adversarial disturbances.

Adversarial Disturbance We implement a bounded policy-based adversarial attack, where at each state x we maximise for the KL divergence between the disturbed and undisturbed state, such that the adversarial operator is:

$$T^{\varepsilon}_{adv}(y \mid x) = 1 \implies y \in \underset{\tilde{x}}{\operatorname{argmax}}\; D_{KL}(\pi(x), \pi(\tilde{x}))$$
$$s.t.\; \|x - \tilde{x}\|_2 \leq \varepsilon.$$

Noise	*PPO on MiniGrid Environments*				*A2C on MiniGrid Environments*			
	Vanilla	$\text{LR}_{\text{PPO}}(K_T^u)$	$\text{LR}_{\text{PPO}}(K_T^g)$	SA-PPO	Vanilla	$\text{LR}_{\text{A2C}}(K_T^u)$	$\text{LR}_{\text{A2C}}(K_T^g)$	$\text{LR}_{\text{A2C}}(K_D)$
LavaGap								
$\emptyset$	**0.95±0.003**	**0.95±0.075**	**0.95±0.101**	0.94±0.068	**0.94±0.004**	**0.94±0.005**	**0.94±0.003**	**0.94±0.006**
T_1	0.80±0.041	**0.95±0.078**	0.93±0.124	0.88±0.064	0.83±0.061	**0.93±0.019**	0.89±0.032	0.91±0.088
T_2	0.92±0.015	**0.95±0.052**	**0.95±0.094**	0.93±0.050	0.89±0.029	**0.94±0.008**	0.93±0.011	0.93±0.021
$T_{adv}^{0.5}$	0.56±0.194	**0.93±0.101**	0.91±0.076	0.90±0.123	0.92±0.034	**0.94±0.003**	**0.94±0.007**	0.93±0.015
T_{adv}^{1}	0.20±0.243	**0.90±0.124**	0.68±0.190	**0.90±0.135**	0.75±0.123	**0.94±0.006**	0.92±0.038	0.88±0.084
T_{adv}^{2}	0.01±0.051	0.71±0.251	0.21±0.357	**0.87±0.116**	0.27±0.119	**0.79±0.069**	0.68±0.127	0.56±0.249
LavaCrossing								
$\emptyset$	**0.95±0.023**	0.93±0.050	0.93±0.018	0.88±0.091	0.91±0.024	0.91±0.063	0.90±0.017	**0.92±0.034**
T_1	0.50±0.110	**0.92±0.053**	0.89±0.029	0.64±0.109	0.66±0.071	**0.78±0.111**	0.72±0.073	0.76±0.098
T_2	0.84±0.061	**0.92±0.050**	**0.92±0.021**	0.85±0.094	0.78±0.054	0.83±0.105	0.86±0.029	**0.87±0.063**
$T_{adv}^{0.5}$	0.29±0.098	**0.91±0.081**	**0.91±0.054**	0.87±0.045	0.56±0.039	0.51±0.089	0.43±0.041	**0.68±0.126**
T_{adv}^{1}	0.03±0.022	0.83±0.122	0.86±0.132	**0.87±0.059**	0.27±0.158	0.25±0.118	0.17±0.067	**0.43±0.060**
T_{adv}^{2}	0.0±0.004	0.50±0.171	0.38±0.020	**0.82±0.072**	0.06±0.056	0.04±0.030	0.01±0.008	**0.09±0.060**
DynamicObstacles								
$\emptyset$	**0.91±0.002**	**0.91±0.008**	**0.91±0.007**	**0.91±0.131**	**0.91±0.011**	0.88±0.020	0.89±0.009	**0.91±0.013**
T_1	0.23±0.201	**0.77±0.102**	0.61±0.119	0.45±0.188	0.27±0.104	0.43±0.108	0.45±0.162	**0.56±0.270**
T_2	0.50±0.117	**0.75±0.075**	0.70±0.072	0.68±0.490	0.45±0.086	0.53±0.109	0.52±0.161	**0.67±0.203**
$T_{adv}^{0.5}$	0.74±0.230	0.89±0.118	0.85±0.061	**0.90±0.142**	0.46±0.214	0.55±0.197	0.51±0.371	**0.62±0.249**
T_{adv}^{1}	0.26±0.269	0.79±0.157	0.68±0.144	**0.84±0.150**	0.19±0.284	**0.35±0.197**	0.23±0.370	0.10±0.379
T_{adv}^{2}	-0.49±0.312	0.51±0.234	0.33±0.202	**0.55±0.170**	-0.54±0.209	-0.21±0.192	-0.53±0.261	**-0.51±0.260**

Table 6: Extended Reward Results.

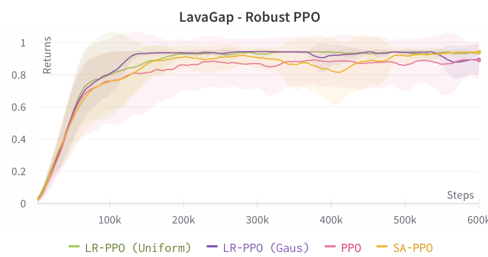
The optimisation problem is solved at every point by using a Stochastic Gradient Langevin Dynamics (SGLD) optimiser. The results are presented in Table 6.

This type of adversarial attack with SGLD optimiser was proposed in Zhang et al. [2020]. As one can see, the adversarial disturbance is quite successful at severely lowering the obtained rewards in all scenarios. Additionally, as expected SA-PPO was the most effective at minimizing the disturbance effect (as it is trained with adversarial disturbances), although LRPG produces reasonably robust policies against this type of disturbances as well. At last, A2C appears to be much more sensitive to adversarial disturbances than PPO, indicating that the policies produced by PPO are by default more robust than A2C.

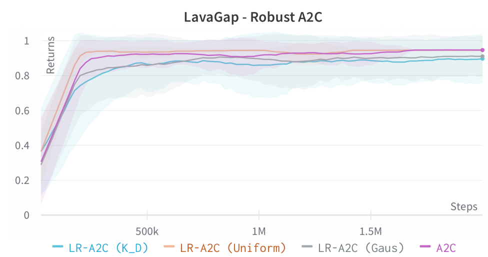
C.2 Continuous Control

The continuous control environments simulated are MountainCar, LunarLander and BipedalWalker. The policies used are in all cases MLP policies with ReLU gates and a $(64, 64)$ feature extractor plus a fully connected layer to output the values and actions unless stated otherwise. The hyperparameters can be found in tables 7 and 8. The implementation is based on Stable Baselines 3 [Raffin et al., 2021] tuned algorithms.

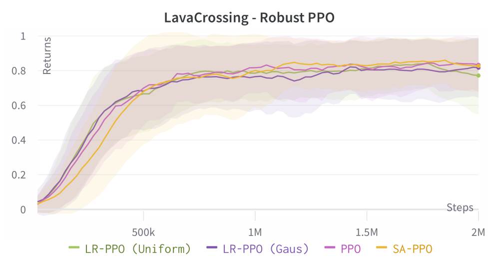
Noise Kernels. We consider again two types of noise; a normal distributed noise $\tilde{T}^g$ and a uniform distributed noise $\tilde{T}^u$. In all cases, algorithms are implemented with a state observation normalizer. That is, assimptotically all states will be observed to be in the set $(-1, 1)$. For this reason, the uniform noise is bounded at lower values than for the discrete control environments. For BipedalWalker $\|\xi\|_\infty \leq 0.05$ and for Lunarlander and MountainCar $\|\xi\|_\infty \leq 0.1$. Larger values were shown to destabilize learning.



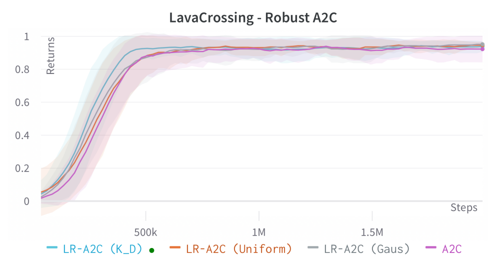
(a) LavaGap with PPO



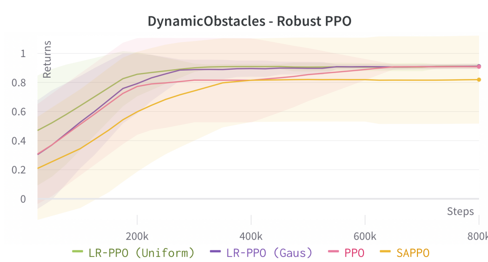
(b) LavaGap with A2C



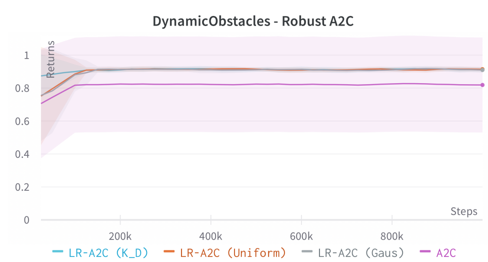
(c) LavaCrossing with PPO



(d) LavaCrossing with A2C



(e) DynamicObstacles with PPO



(f) DynamicObstacles with SAC

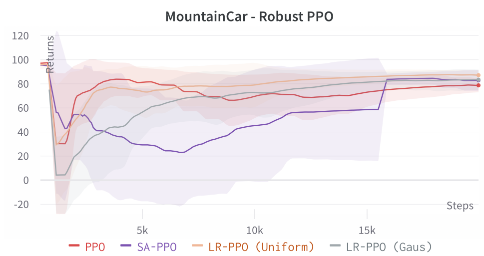
Figure 4: Learning Plots for Discrete Control Environments.

	MountainCarContinuous	LunarLanderContinuous	BipedalWalker-v3
Parallel Envs	1	16	32
Steps	2×10^4	1×10^6	5×10^6
γ	0.9999	0.999	0.999
α	3×10^{-4}	3×10^{-4}	3×10^{-4}
Grad. Clip	5	0.5	0.5
Ratio Clip	0.2	0.2	0.18
Gae	0.9	0.98	0.95
Epochs	10	4	10
E. Coeff	0.00429	0.01	0

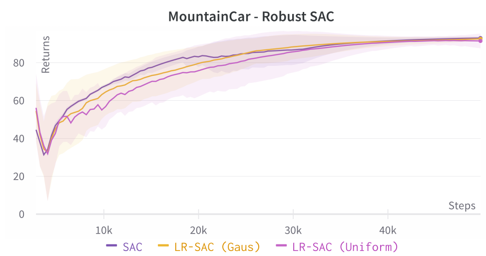
Table 7: PPO Parameters for Continuous Control

	MountainCarContinuous	LunarLanderContinuous	BipedalWalker-v3
Steps	5×10^4	5×10^5	5×10^5
γ	0.9999	0.99	0.98
α	3×10^{-4}	7.3×10^{-4}	7.3×10^{-4}
τ	0.01	0.01	0.01
Train Freq.	32	1	64
Grad. Steps	32	1	64
MLP Arch	(64,64)	(400,300)	(400,300)

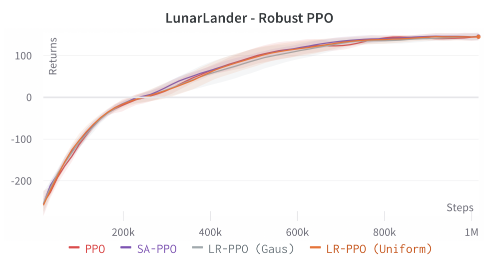
Table 8: SAC Parameters for Continuous Control



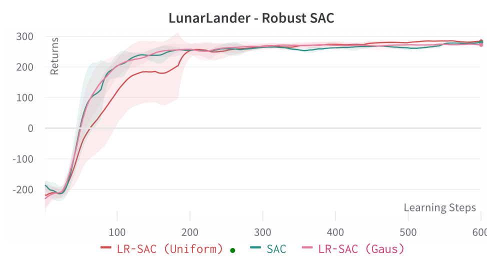
(a) MountainCar with PPO



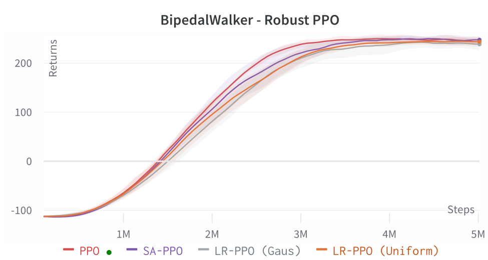
(b) MountainCar with SAC



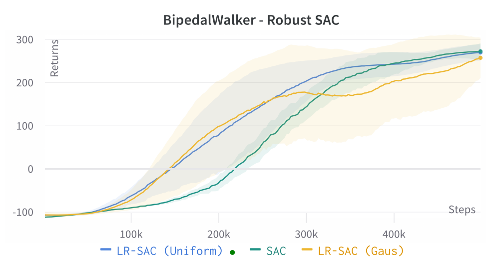
(c) LunarLander with PPO



(d) LunarLander with SAC



(e) BipedalWalker with PPO



(f) BipedalWalker with SAC

Figure 5: Learning Plots for Continuous Control Environments.

Learning processes. In general, learning was not severlely affected by the LRPG scheme. However, it was shown to induce a larger variance in the trajectories observed, as seen in LunarLander with LR-SAC and BipedalWalker with LR-SAC.